# Eigenvalue decomposition of a symmetric matrix over the symmetrized max-plus algebra

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## ARTICLE INFO

## ABSTRACT

*This paper discusses topics in the symmetrized max-plus algebra. In this study, it will be shown the existence of eigenvalue decomposition of a symmetric matrix over symmetrized max-plus algebra. Eigenvalue decomposition is shown by using a function that corresponds to the symmetrized max-plus algebra with conventional algebra. The result obtained is the existence of eigenvalue decomposition of a symmetric matrix over symmetrized max-plus algebra and its application to determine eigenvalues and eigenvectors.*

http://ejournal.radenintan.ac.id/index.php/desimal/index

## INTRODUCTION

In linear algebra (conventional algebra), eigenvalues are characteristic values of a matrix of size $n \times n$, while eigenvectors are non-zero column vectors when multiplied by a matrix of size $n \times n$ will produce another vector that has multiple values of the eigenvector itself. There have been many discussions about eigenvalues and eigenvectors in real matrices (Kuttler, 2012; Lay et al., 2016). The eigenvalue decomposition is one of the matrix decompositions based on the eigenvalue of the matrix.

The max-plus algebra is the set of all reals $\mathbb{R} \cup \{-\infty\}$ with maximum (written with "max") as addition and common sum (written with "plus") as multiplication, and it is denoted by $\mathbb{R}_{max}$. The main difference between the max-plus algebra and conventional algebra is its additive inverse. There is no additive inverse for all elements in the max-plus algebra, except for zero elements (Hogben, 2014). There have been many discussions about eigenvalue and eigenvectors in the max-plus matrix (De Schutter et al., 2020; James, 2016; Yonggu & Hee, 2018).

The symmetrization process of the max-plus algebra can be done to get the minus and balance of elements in $\mathbb{R}_{max}$ (Leake et al., 1994). The symmetrization of $\mathbb{R}_{max}$ is called the symmetrized max-plus algebra and denoted by $\mathbb{S}$. Furthermore, $\mathbb{R}_{max}$ can be viewed as a positive or zero part of $\mathbb{S}$. The $QR$ and singular value decomposition of the symmetrized max-plus algebraic matrix was discussed in De Schutter & De Moor (2002). The $LU$-decomposition of the symmetrized max-plus algebraic matrix was discussed in Suroto et al. (2018). The function which corresponds to symmetrized max-plus algebra and conventional algebra is used as a link to solve symmetrized max-plus problems in the conventional algebra sense.

The discussion of the eigenvalue of symmetrized max-plus algebra has been carried out by Ariyanti et al. (2015). The eigenvalue of the symmetrized max-plus algebraic matrix was determined using an extended linear complementary problem (ELCP). The eigenvalues obtained by ELCP cannot be performed in a similar way as in conventional algebra, and it is one of the disadvantages of this method. In Ariyanti (2021) also discusses eigenvalue problems in the symmetrized max-plus algebra. It illustrates the necessity or sufficient of eigenvalue in matrix over $\mathbb{S}$, but the technique for calculating eigenvalue and eigenvectors is not illustrated. Meanwhile, in discussing the eigenvalue in matrix over $\mathbb{S}$, the main difficulty is the technique of calculating eigenvalue and eigenvectors which are not illustrated in those articles.

This paper discusses the eigenvalue decomposition of a matrix over the symmetrized max-plus algebra. It will be used to determine the eigenvalue and eigenvectors of a matrix over $\mathbb{S}$. The matrix discussed in this paper is a symmetric matrix. This is due to the use of a given rotation which will be applied when zeroing entries other than the main diagonal when decomposites the matrix. This can be done when the matrix is a symmetric matrix. We also use a function in De Schutter & De Moor (2002) to correspond the symmetrized max-plus algebra with conventional algebra.

The results obtained in this paper have the advantage that the completion of eigenvalue and eigenvectors of the symmetrized max-plus algebraic matrix can be done as in a conventional matrix. The results in this paper can potentially be developed for any matrix (not necessarily a symmetric matrix) over symmetrized max-plus algebra.

## METHOD

This research is a literature study that is developing research that already exists, namely matrix decompositions. This study develops the eigenvalues decomposition of the symmetrized max-plus algebraic matrix to determine eigenvalues and eigenvectors of a symmetric matrix over $\mathbb{S}$ as in conventional algebra. This improves discussion in Ariyanti et al. (2015) and Ariyanti (2021) when determining eigenvalues and eigenvectors of a symmetric matrix.

A function that corresponds to the symmetric max-plus algebra with conventional algebra in De Schutter & De Moor (2002) is used to determine eigenvalues decomposition. The problems solving of the existence of eigenvalues decomposition are done in conventional algebra sense. The research steps are

1. Literature review in some topics i.e., eigenvalues and eigenvectors in conventional algebra, eigenvalue decomposition in conventional algebra, max-plus algebra, and symmetrized max-plus algebra.
2. Define eigenvalues and eigenvectors in the symmetrized max-plus algebraic matrix.
3. Determine the existence of the eigenvalue decomposition in

symmetric matrix over the symmetrized max-plus algebra.
4. Determine eigenvalues and eigenvectors of a symmetric matrix using eigenvalue decomposition.

## RESULTS AND DISCUSSION

This section is the main result of this research which is to determine the eigenvalue of a symmetric matrix using eigenvalue decomposition. At the beginning of the discussion, the eigenvalue in the matrix over the symmetrized max-plus algebra is first defined. The following definition is done adopting the definition of eigenvalue in conventional algebra (Lay et al., 2016). The "balance relation" in symmetrized max-plus algebra (Leake et al., 1994) plays the role of an "equal relation" in conventional algebra.

### Definition 1
*An eigenvector of a matrix $A \in \mathbb{S}^{n \times n}$ is nonzero vector $\bar{\mathrm{x}} \in (\mathbb{S}^{\vee})^{n \times n}$ such that*

$$A \otimes \bar{\mathrm{x}} \nabla \lambda \otimes \bar{\mathrm{x}} \qquad (1)$$

*for some scalar $\lambda \in \mathbb{S}^{\vee}$. A scalar $\lambda$ is called an eigenvalue of A if there is a nontrivial solution $\bar{\mathrm{x}}$ of (1); such an $\bar{\mathrm{x}}$ is called an eigenvector corresponding to $\lambda$.*

The following is an example to shows the eigenvalue in Definition 1

### Example 2
Suppose $A = \begin{bmatrix} 1 & 2^{\bullet} \\ 4 & 1 \end{bmatrix}$ and a nonzero vector $\bar{\mathrm{x}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that

$$A \otimes \bar{\mathrm{x}} = \begin{bmatrix} 1 & 2^{\bullet} \\ 4 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^{\bullet} \\ 4 \end{bmatrix}$$

$$\nabla \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \otimes \bar{\mathrm{x}}.$$

Since $A \otimes \bar{\mathrm{x}} \nabla 3 \otimes \bar{\mathrm{x}}$ then 3 is eigenvalue of $A$ and a nonzero vector $\bar{\mathrm{x}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigenvector of $A$ corresponding to eigenvalue 3. ∎

Based on the balance (1), it is obtained

$$(A \ominus \lambda \otimes I) \otimes \bar{\mathrm{x}} \nabla \bar{\mathcal{E}}. \qquad (2)$$

According to the theorem of linear balance systems in Ariyanti (2021), then (2) has a non-trivial solution if and only if $\det(A \ominus \lambda \otimes I) \nabla \mathcal{E}$.

In conventional algebra, the discussion of the eigenvalue of a matrix is identical to the characteristic equation. Furthermore, the eigenvalue can be determined by calculating the roots of the characteristic equation. The following is a definition of characteristic balance on a symmetrized max-plus algebraic matrix.

### Definition 3
*The characteristic balance of A is*

$$\det(A \ominus \lambda \otimes I) \nabla \mathcal{E}. \qquad (3)$$

*Furthermore, the scalars $\lambda$ that satisfies (3) are called eigenvalue of A.*

The following is an example to illustrate the characteristic balance of a symmetrized max-plus algebraic matrix.

### Example 4
Suppose $A = \begin{bmatrix} 1 & 2^{\bullet} \\ 4 & 1 \end{bmatrix}$. The characteristic balance of $A$ is

$$\det(A \ominus \lambda \otimes I) \nabla \mathcal{E}$$

and it is obtained

$$\lambda^2 \ominus 1 \otimes \lambda \ominus 6^{\bullet} \nabla \mathcal{E} \blacksquare$$

It should be noted that $\det(A \ominus \lambda \otimes I)$ is a polynomial over symmetrized max-plus algebra. In the discussion of symmetrized max-plus algebra, there has never been a special discussion that illustrates the roots of a polynomial. Thus, this paper will not use the characteristic balance to determine the eigenvalue of a matrix. In this paper, the eigenvalue decomposition is used to determine the eigenvalue of the symmetric matrix over $\mathbb{S}$.

The existence of eigenvalue decomposition in symmetrized max-plus algebra is done by adopting eigenvalue decomposition in conventional algebra in Anton & Rorres (2010).

The following theorem illustrates the existence of eigenvalue decomposition in the symmetric matrix over symmetrized max-plus algebra.

**Theorem 5**

*If $A \in \mathbb{S}^{n \times n}$ is symmetric then there are $U \in (\mathbb{S}^{\vee})^{n \times n}$ and $D \in (\mathbb{S}^{\vee})^{n \times n}$ with $D$ is a diagonal matrix such that*

$$A \nabla U \otimes D \otimes U^T$$

*where $U^T \otimes U \nabla I_n$*

**Proof.**

Let $A \in \mathbb{S}^{n \times n}$ has non signed entries, then it is defined $\hat{A} \in (\mathbb{S}^{\vee})^{n \times n}$ such that

$$\hat{a}_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} \text{ signed} \\ |a_{ij}|_{\oplus}, & \text{if } a_{ij} \text{ non signed} \end{cases}$$

for all $i, j$. So, in this paper, it is only proven for all signed matrix. Assume that $A$ is a signed matrix. If $A = \mathcal{E}_{n \times n}$ then there are $U = I_n$ and $D = \mathcal{E}_{n \times n}$ such that

$$I_n \otimes \mathcal{E}_{n \times n} \otimes (I_n)^T = A \nabla A$$

and

$$(I_n)^T \otimes I_n = I_n \nabla I_n.$$

Next, the proof is carried out for the symmetric matrix $A \neq \mathcal{E}_{n \times n} \in \mathbb{S}^{n \times n}$. By using function $\mathcal{F}$, let $\tilde{A} = \mathcal{F}(A, M, \cdot)$ with $M \in \mathbb{R}_0^{n \times n}$, $m_{ij} \in \{-1, 1\}$ for all $i, j$. Therefore, entries of $\tilde{A}$ i.e $\tilde{a}_{ij}(s) = m_{ij} e^{|a_{ij}|_{\oplus} s}$ are element in $S_e$. The function $\mathcal{F}$ and definition of $S_e$ refer to De Schutter & De Moor (2002).

The Jacobian method in conventional algebra (Golub & Loan, 2013) is used to make zero the off-diagonal entries in matrix over $S_e$. Let

$$\begin{bmatrix} \tilde{x}(s) & \tilde{w}(s) \\ \tilde{w}(s) & \tilde{y}(s) \end{bmatrix}$$

is a symmetric matrix in $S_e$, then the off-diagonal entries $\tilde{w}(s)$ would be zero by Jacobi method.

$$\begin{bmatrix} \hat{x} & 0 \\ 0 & \hat{y} \end{bmatrix} = \begin{bmatrix} c & t \\ -t & c \end{bmatrix}^T \begin{bmatrix} \tilde{x}(s) & \tilde{w}(s) \\ \tilde{w}(s) & \tilde{y}(s) \end{bmatrix} \begin{bmatrix} c & t \\ -t & c \end{bmatrix}$$

where

$$\alpha = \frac{\tilde{y}(s) - \tilde{x}(s)}{2\tilde{w}(s)},$$

$$\tau = \frac{sign(\alpha)}{|\alpha| + \sqrt{1 + \alpha^2}}$$

and

$$c = \frac{1}{\sqrt{1 + \tau^2}}, t = \tau c.$$

This step is done repeatedly in the off-diagonal entry to get the diagonal form for $\tilde{A}$.

Since the Jacobi method is applied to symmetric matrix $\tilde{A}$, then it will transform $\tilde{A}$ to diagonal form $\tilde{D}$ and yields Jacobian transformation series

$$\tilde{A}^{(k+1)} = J^T \tilde{A}^{(k)} J$$

where $\tilde{A}^{(0)} = \tilde{A}$ and $\lim_{k \to \infty} \tilde{A}^{(k)} = \tilde{D}$ is diagonal form. For every Jacobian transformation, we selected such that make zero the off-diagonal entries $\tilde{a}_{ij}^{(k)} = \tilde{a}_{ji}^{(k)}$ that have maximal absolute value. By using the Frobenius norm, the Jacobian method is quadratic convergent. It will make zero the off-diagonal by increasing the norm of diagonal entries and decreasing the norm of off-diagonal entries. Therefore, the Jacobian transformation will transform $\tilde{A}$ to diagonal form $\tilde{D}$.

Let $\tilde{U} \tilde{D} \tilde{U}^T$ approximates path eigenvalue decomposition of $\tilde{A}$ in $[L, \infty)$ by Jacobian transformation. Since all of entries in $\tilde{U}$, $\tilde{D}$ and $\tilde{U}^T$ are element in $S_e$ then

$$\mathcal{F}(A, N, s) \sim \tilde{U}(s) \tilde{D}(s) \tilde{U}^T(s)$$

$s \to \infty$ for some $N \in \mathbb{R}_0^{m \times n}$. Furthermore, $\tilde{U}^T(s) \tilde{U}(s) \sim I_n$, for $s \geq L$. The diagonal entries of $\tilde{D}$ and all of entries of $\tilde{U}$ are element in $S_e$, and will asymtotically equivalent with an exponential in the neighborhood of $\infty$.

Let $\tilde{\sigma}_i = \tilde{D}_{ii}$ for $i = 1, 2, \dots, n$. By applying the reverse function $\mathcal{R}$ and link between symmetrized max-plus algebra

and conventional matrix in De Schutter & De Moor (2002), it is obtained $D = \mathcal{R}(\widetilde{D})$, $U = \mathcal{R}(\widetilde{U})$ and $\sigma_i = (D)_{ii} = \mathcal{R}(\widetilde{\sigma}_i)$ for all $i$. Matrix $D$ is a diagonal matrix with signed entries and $U$ also has signed entries. Therefore, it is obtained that

$$A \, \nabla \, U \otimes D \otimes U^T$$

with $U^T \otimes U \, \nabla I_n$. ∎

The following is an example to illustrate the existence of eigenvalue decomposition in Theorem 5.

**Example 6**
Suppose a symmetric matrix $A = \begin{bmatrix} 8 & \ominus 9 \\ \ominus 9 & 10^{\bullet} \end{bmatrix}$. Then, we will determine the eigenvalue decomposition of $A$. In the beginning, it is defined $\tilde{A} = \mathcal{F}(A, M, \cdot)$ where $M \in \mathbb{R}_0^{2 \times 2}$ and $m_{ij} \in \{-1, 1\}$ for all $i, j$ it is obtained

$$\tilde{A}(s) = \begin{bmatrix} e^{8s} & -e^{9s} \\ -e^{9s} & e^{10s} \end{bmatrix}$$

for all $s \in \mathbb{R}_0^+$.

The given rotations formula, i.e

$$\begin{bmatrix} \hat{x} & 0 \\ 0 & \hat{y} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}^T \begin{bmatrix} x & w \\ w & y \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}.$$

can be used to determine the eigenvalue decomposition of $\tilde{A}$.

$$\tilde{A}(s) \sim \widetilde{U}(s)\widetilde{D}(s)\widetilde{U}(s)^T$$

where

$$\widetilde{U}(s) = \begin{bmatrix} 1 & -e^{-s} \\ e^{-s} & 1 \end{bmatrix}$$

$$\widetilde{D}(s) = \begin{bmatrix} 0 & 0 \\ 0 & e^{10s} \end{bmatrix}$$

$$\tilde{A}(s) = \begin{bmatrix} e^{8s} & -e^{9s} \\ -e^{9s} & e^{10s} \end{bmatrix}.$$

By using reverse map $\mathcal{R}$ and link between symmetrized max-plus algebra and conventional algebra, then it is obtained

$$U = \mathcal{R}(\widetilde{U}) = \begin{bmatrix} 0 & \ominus -1 \\ -1 & 0 \end{bmatrix}$$

$$A = \mathcal{R}(\tilde{A}) = \begin{bmatrix} 8 & \ominus 9 \\ \ominus 9 & 10 \end{bmatrix}$$

$$D = \mathcal{R}(\widetilde{D}) = \begin{bmatrix} \mathcal{E} & \mathcal{E} \\ \mathcal{E} & 10 \end{bmatrix}.$$

Note that

$$U \otimes D \otimes U^T = \begin{bmatrix} 8 & \ominus 9 \\ \ominus 9 & 10 \end{bmatrix} \nabla A$$

$$U^T \otimes U = \begin{bmatrix} 0 & (-1)^{\bullet} \\ (-1)^{\bullet} & 0 \end{bmatrix} \nabla I_2$$

$$U \otimes U^T = \begin{bmatrix} 0 & (-1)^{\bullet} \\ (-1)^{\bullet} & 0 \end{bmatrix} \nabla I_2.$$

For $A = \begin{bmatrix} 8 & \ominus 9 \\ \ominus 9 & 10^{\bullet} \end{bmatrix} \in \mathbb{S}^{n \times n}$, there are a diagonal matrix $D = \begin{bmatrix} \mathcal{E} & \mathcal{E} \\ \mathcal{E} & 10 \end{bmatrix}$ and matrix $U = \begin{bmatrix} 0 & \ominus -1 \\ -1 & 0 \end{bmatrix}$ such that

$$A \, \nabla \, U \otimes D \otimes U^T$$

where $U \otimes U^T \nabla I_2$ dan $U^T \otimes U \, \nabla I_2$ ∎

Note that in Theorem 5, the eigenvalue decomposition of $A$ is

$$A \, \nabla \, U \otimes D \otimes U^T$$

where $U^T \otimes U \, \nabla I_n$. There are many possibilities entries in $U^T \otimes U$:

1. Each entry of $U^T \otimes U$ are a balanced element.

$$U^T \otimes U = \begin{bmatrix} ( \ )^{\bullet} & ( \ )^{\bullet} & \cdots & ( \ )^{\bullet} \\ ( \ )^{\bullet} & ( \ )^{\bullet} & \cdots & ( \ )^{\bullet} \\ \vdots & \vdots & \ddots & \vdots \\ ( \ )^{\bullet} & ( \ )^{\bullet} & \cdots & ( \ )^{\bullet} \end{bmatrix}$$

2. Each of non-diagonal entries in $U^T \otimes U$ is a balanced element and diagonal entries are 0 or balanced elements.

$$U^T \otimes U = \begin{bmatrix} 0 & ( \ )^{\bullet} & \cdots & ( \ )^{\bullet} \\ ( \ )^{\bullet} & 0 & \cdots & ( \ )^{\bullet} \\ \vdots & \vdots & \ddots & \vdots \\ ( \ )^{\bullet} & ( \ )^{\bullet} & \cdots & ( \ )^{\bullet} \end{bmatrix}$$

3. Each of non-diagonal entries in $U^T \otimes U$ is a balanced element and diagonal entries are 0.

$$U^T \otimes U = \begin{bmatrix} 0 & ( \ )^{\bullet} & \cdots & ( \ )^{\bullet} \\ ( \ )^{\bullet} & 0 & \cdots & ( \ )^{\bullet} \\ \vdots & \vdots & \ddots & \vdots \\ ( \ )^{\bullet} & ( \ )^{\bullet} & \cdots & 0 \end{bmatrix}.$$

Next, consider the balance

$$A \otimes U \nabla U \otimes D. \qquad (4)$$

From the right-hand side of balance (4), it is obtained $U \otimes D$

$$
= \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix} \otimes
$$

$$
\begin{bmatrix} d_{11} & \mathcal{E} & \dots & \mathcal{E} \\ \mathcal{E} & d_{22} & \dots & \mathcal{E} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \dots & d_{nn} \end{bmatrix}
$$

$$
= d_{11} \otimes \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} \oplus \dots \oplus d_{nn} \otimes \begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{bmatrix} \quad (5)
$$

From (4) and (5), it is obtained

$$
A \otimes \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix} \nabla d_{11} \otimes \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix}
$$

$$\vdots \qquad (6)$$

$$
A \otimes \begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{bmatrix} \nabla d_{nn} \otimes \begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{bmatrix}.
$$

All the balances in (6) shows that $d_{11}$, $d_{22}$, ..., $d_{nn}$ are eigenvalues of $A$ and eigenvectors corresponding to $d_{11}$, $d_{22}$, ..., $d_{nn}$ are

$$
\begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ u_{n1} \end{bmatrix}, \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ u_{n2} \end{bmatrix}, \dots, \begin{bmatrix} u_{1n} \\ u_{2n} \\ \vdots \\ u_{nn} \end{bmatrix},
$$

respectively.

In Definition 1, a nonzero vector $\bar{x} \in (\mathbb{S}^{\vee})^{n \times n}$ is called eigenvectors of $A$ if

$$A \otimes \bar{x} \nabla \lambda \otimes \bar{x}. \qquad (7)$$

for a scalar $\lambda \in \mathbb{S}^{\vee}$. The balance (7) can be expressed as

$$A \otimes \bar{x} \nabla \lambda \otimes \bar{x} \oplus \bar{\mathcal{E}}. \qquad (8)$$

Since each of the balanced vectors $\bar{t}^{\bullet} \in (\mathbb{S}^{\bullet})^n$ satisfy $\bar{\mathcal{E}} \nabla \bar{t}^{\bullet}$ where $\bar{\mathcal{E}}$ is a signed

vector, then vector $\bar{\mathcal{E}}$ in (8) can be substituted by $\bar{t}^{\bullet}$, such that

$$A \otimes \bar{x} \nabla \lambda \otimes \bar{x} \oplus \bar{t}^{\bullet}. \qquad (9)$$

Henceforth, the eigenvalue and eigenvectors in the matrix over $\mathbb{S}^{\bullet}$ can be modified by the balance (9), which is presented in the following lemma.

**Lemma 7**

*If $\lambda$ is eigenvalue of $A$ and a nonzero vector $\bar{x}$ is eigenvector corresponding to $\lambda$ then*

$$A \otimes \bar{x} \nabla \lambda \otimes \bar{x} \oplus \bar{t}^{\bullet}$$

*for $\bar{t}^{\bullet}$ is a balanced vector.*

**Proof.**

Suppose $\lambda$ is the eigenvalue of $A$ and a nonzero vector $\bar{x}$ is eigenvector corresponding to $\lambda$. By Definition 1, it is obtained that

$$A \otimes \bar{x} \nabla \lambda \otimes \bar{x}.$$

Since $A \otimes \bar{x} \nabla \lambda \otimes \bar{x}$ can be expressed as

$$A \otimes \bar{x} \nabla \lambda \otimes \bar{x} \oplus \bar{\mathcal{E}}$$

where $\bar{\mathcal{E}}$ is a signed vector, then $\bar{\mathcal{E}}$ can be substituted by any balanced vectors $\bar{t}^{\bullet}$. So, it is obtained

$$A \otimes \bar{x} \nabla \lambda \otimes \bar{x} \oplus \bar{t}^{\bullet}$$

where $\bar{t}^{\bullet}$ is balanced vectors. ∎

The following is an example to illustrate Lemma 7.

**Example 8**

Suppose $A = \begin{bmatrix} 1 & 2^{\bullet} \\ 4 & 1 \end{bmatrix}$ and a nonzero vector $\bar{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $A \otimes \bar{x} = \begin{bmatrix} 3^{\bullet} \\ 4 \end{bmatrix} \nabla \begin{bmatrix} 3 \\ 4 \end{bmatrix} \oplus \begin{bmatrix} t_1^{\bullet} \\ t_2^{\bullet} \end{bmatrix} = 3 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus \begin{bmatrix} t_1^{\bullet} \\ t_2^{\bullet} \end{bmatrix} = 3 \otimes \bar{x} \oplus \begin{bmatrix} t_1^{\bullet} \\ t_2^{\bullet} \end{bmatrix}.$ ∎

Lemma 28 can also be applied in a symmetric matrix. From the existence of eigenvalue decomposition of a symmetric matrix in Theorem 26.

$$A \nabla U \otimes D \otimes U^T$$

then one gets

$$A \otimes U \nabla U \otimes D \otimes U^T \otimes U. \quad (10)$$

The multiplication in both side of balance in (10) results

$$A \otimes [U]_1 \nabla d_{11} \otimes [U]_1 \oplus \bar{t}_1^\bullet$$

$$A \otimes [U]_2 \nabla d_{22} \otimes [U]_2 \oplus \bar{t}_2^\bullet$$

$$\vdots$$

$$A \otimes [U]_n \nabla d_{nn} \otimes [U]_n \oplus \bar{t}_n^\bullet \qquad (11)$$

where $[U]_1, [U]_2, \dots, [U]_n$ are the first, second, ..., $n$-th columns of $U$, respectively, $d_{11}, d_{22}, \dots, d_{nn}$ are diagonal entries of $D$, respectively, and $\bar{t}_1^\bullet, \bar{t}_2^\bullet, \dots, \bar{t}_n^\bullet$ are the $n$-balanced vectors. From (11), one can use eigenvalue decomposition of the symmetric matrix in the symmetrized max-plus algebra to determine eigenvalue of the symmetric matrix.

The following is an example to illustrate that the existence of eigenvalue decomposition in Theorem 3 can be used to determine the eigenvalue of a matrix.

**Example 9**

Suppose a symmetric matrix $A = \begin{bmatrix} 8 & \ominus 9 \\ \ominus 9 & 10^\bullet \end{bmatrix}$. Eigenvalue decomposition of $A$ is

$$A \nabla U \otimes D \otimes U^T$$

for $U = \begin{bmatrix} 0 & \ominus -1 \\ -1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} \mathcal{E} & \mathcal{E} \\ \mathcal{E} & 10 \end{bmatrix}$. Let $d_{11} = \mathcal{E}$, $d_{22} = 10$, $[U]_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $[U]_2 = \begin{bmatrix} \ominus -1 \\ 0 \end{bmatrix}$. Note that

$$A \otimes [U]_1 = \begin{bmatrix} 8^\bullet \\ 9^\bullet \end{bmatrix} \nabla \begin{bmatrix} \mathcal{E} \\ \mathcal{E} \end{bmatrix} = d_{11} \otimes [U]_1$$

$$A \otimes [U]_2 = \begin{bmatrix} \ominus 9 \\ 10^\bullet \end{bmatrix} \nabla \begin{bmatrix} \ominus 9 \\ 10 \end{bmatrix} = d_{22} \otimes [U]_2.$$

So, eigenvalues of $A$ are $d_{11} = \mathcal{E}$ and $d_{22} = 10$. Then, nonzero vectors $[U]_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $[U]_2 = \begin{bmatrix} \ominus -1 \\ 0 \end{bmatrix}$ are eigenvectors corresponding to $d_{11}$ and $d_{22}$, respectively. ∎

**CONCLUSIONS AND SUGGESTIONS**

The existence of eigenvalue decomposition of the symmetric matrix over symmetrized max-plus algebra can be investigated using a link among the symmetrized max-plus algebra and conventional algebra. The determination of eigenvalue decomposition can be done similarly to conventional algebra. Furthermore, this decomposition can be used to determine the eigenvalues and eigenvectors of a symmetric matrix over symmetrized max-plus algebra.

Future research can be potentially done in diagonalization problems of symmetrized max-plus algebraic matrix.

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