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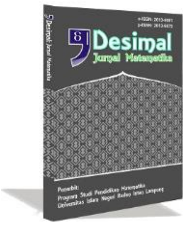

Variational homotopy perturbation method for solving systems of homogeneous linear and nonlinear partial differential equations

Atika Faradilla, Aang Nuryaman*, Asmiati, Dewi Rakhmatia Nur

Lampung University, Indonesia

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*Correspondence: E-mail:
aang.nuryaman@fmipa.unila.ac.id

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ABSTRACT

The variational homotopy perturbation method is developed by combining variational iteration method and homotopy perturbation method. Variational iteration method has an efficient process in solving a wide variety of equations and systems of equations. Meanwhile, homotopy perturbation method gives a solution series which in most cases converge very quickly. The developed method, variational homotopy perturbation method, took full advantage of both methods. In this study, we described an application of the variational homotopy perturbation method to solve systems of homogeneous partial differential equations. Here we consider some initial value problems of homogeneous partial differential equation systems with two and three variables. The results show that the obtained solution using this method was in agreement with the solution using the homotopy analysis method and variational iteration method, which prove the validity of the variational homotopy perturbation method when applied to systems of partial differential equations.

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INTRODUCTION

A partial differential equation can be encountered when a certain problem is related to physics and geometry if the function related is dependent on two or more than two variables. Complicated problem, such as problem in advanced physics, needs to be modelled into partial differential equation. But these partial differential equations are difficult to get their exact solutions, especially for the nonlinear ones. Nowadays, a lot of method have been developed to solve partial differential equation, whether it is an analytic or numerical method.

The homotopy perturbation method was first suggested by He (1999a). In many cases, this method provides a solution series with very fast convergence. Generally, one iteration has provided a high level of the solution accuracy (Matinfar & Saeidy, 2009). Some applications of this method can be found in (He, 2005), (Biazar et al., 2007), (Biazar, 2008), (Biazar et al., 2009), (Matinfar & Saeidy, 2009), (Gepreel, 2011), (Nofel, 2014), (Wu & He, 2018) (Nuryaman, 2019) and others. Based on the previous researches, this method is very effective and simple.

Another method called variational iteration method was also first suggested by He (1999b). This method gives rapidly convergent successive approximations of the exact solution. In addition, this method could reduce the calculation size while keep the high accuracy of the numerical solution (Wazwaz, 2007). Various applications in sciences and engineering used this method and were claimed to be more efficient than adomian decomposition method. Some researchers have reported the application of variational iteration method, for examples: Jin (2009), Ateş & Yildirim (2009), Akbarzade & Langari (2011), Shang & Han (2010), and Olayiwola (2016).

The variational homotopy perturbation method has suggested blending the homotopy perturbation method and the variational iteration method. It took full advantage of both methods. This method is put in without discretization, confining assumptions, or transformations and is free from rounding errors (Noor & Mohyud-Din, 2008). In contrast to the variable separation method which needs boundary conditions and initial conditions, the variational homotopy method only needs the initial conditions to get an analytical solution. This method works efficiently and the results are very reliable.

The variational homotopy perturbation method had been used to solve a lot of scientific problems in recent years. Matinfar et al. (2010) have used this method for ordinary differential equations (ODE's). The application of this method to partial differential equations (PDE's) that have reported, for examples, are reaction-diffusion-convection problems (Daga & Pradhan, 2013), Burgers equation (Hendi et al., 2013), partial differential problem in fluid mechanics (Allahviranloo et al., 2014), and Fisher equation (Matinfar et al., 2010). The solved problems in all these articles are a partial differential equation. However, there are no studies that apply this method in solving systems of partial differential equations. So, in this study, an alternative approach is given based on the variational homotopy perturbation method to solve a system of PDE's. Specifically, systems of homogeneous linear and nonlinear PDE's with two and three variables.

METHOD

In this section, some theories used in this research will be presented.

Homotopy Perturbation Method

Consider an equation in general form,

$$L(u) = 0, \tag{1}$$

with L is any operator of differential or integral. Define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u). \tag{2}$$

Here, $F(u)$ is an operator of functional with known solutions v_0 that can be acquired easily. It is obvious that, for the equation $H(u,p) = 0$ gives

$$H(u,0) = F(u), \quad H(u,1) = L(u). \tag{3}$$

This suggests that $H(u,p)$ traced continuously a curve that is defined implicitly from an initial point $H(v_0,0)$ to a function of solution $H(f,1)$. The embedding parameter $p \in [0,1]$ can be put as a parameter of expanding. The homotopy parameter p was used by the homotopy perturbation method as an embedding parameter to get

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \quad (4)$$

For $p \to 1$, we have (4) becomes

$$f = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i. \quad (5)$$

It is well known that, for most cases, (4) is convergent series with the convergence rate depending on $L(u)$. Furthermore, the limit in (5) is assumed to have a unique solution. Comparisons of similar powers of p gives various order solutions.

Variational Iteration Method

Consider the system of PDE's in an operator form as follow:

$$L_t(u) + R_1(u,v,w) + N_1(u,v,w) = g_1,$$
$$L_t(v) + R_2(u,v,w) + N_2(u,v,w) = g_2,$$
$$L_t(w) + R_3(u,v,w) + N_3(u,v,w) = g_3, \quad (6)$$

The initial conditions are given by

$$u(x,0) = f_1(x),$$
$$v(x,0) = f_2(x),$$
$$w(x,0) = f_3(x), \quad (7)$$

where L_t is a differential operator, R_j and N_j, $j = 1,2,3$, respectively are linear and nonlinear operator, and g_1, g_2, g_3 are the source terms. Under variational iteration method, we can produce a correct functional for system of equation as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(Lu_n(\xi) + R_1(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_1(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_1(\xi))\, d\xi,$$
$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(Lu_n(\xi) + R_2(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_2(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_2(\xi))\, d\xi,$$
$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3(Lu_n(\xi) + R_3(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_3(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_3(\xi))\, d\xi, \quad (8)$$

where λ_j, $1 \le j \le 3$ are Lagrange multipliers. Determination of the appropriate Lagrange multiplier λ_j can be done optimally through variational iteration method. The initial step is to find the Lagrange multiplier λ optimally. The next approximation $u_{n+1}(x,t)$, $v_{n+1}(x,t)$, $w_{n+1}(x,t), n \ge 0$ of the solution $u(x,t), v(x,t), w(x,t)$ will be immediately acquired upon using the determined Lagrange multipliers and any selective functions u_0, v_0, w_0. As a result, we have the solutions are

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),$$
$$v(x,t) = \lim_{n\to\infty} v_n(x,t),$$
$$w(x,t) = \lim_{n\to\infty} w_n(x,t). \quad (9)$$

Variational Homotopy Perturbation Method

Consider system of PDE's in an operator form as follow:

$$L_t(u) + R_1(u,v,w) + N_1(u,v,w) = g_1,$$
$$L_t(v) + R_2(u,v,w) + N_2(u,v,w) = g_2,$$
$$L_t(w) + R_3(u,v,w) + N_3(u,v,w) = g_3, \quad (10)$$

where L_t is a differential operator, R_j and N_j, $j = 1,2,3$, respectively are linear and nonlinear operator, and g_1, g_2, g_3 are the source term. By using the same procedure as before, we can construct a correct functional for system of equation as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(Lu_n(\xi) + R_1(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_1(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_1(\xi))\, d\xi,$$
$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(Lu_n(\xi) + R_2(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_2(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_2(\xi))\, d\xi,$$
$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_3(Lu_n(\xi) + R_3(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_3(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_3(\xi))\, d\xi, \quad (11)$$

where λ_j, $1 \le j \le 3$ are Lagrange multipliers, which can be identified optimally via variational iteration method. Next, we adjus the homotopy perturbation method,

$$\sum_{n=0}^{\infty} p^{(n)} u_n(x,t) = u_0(x,t) + p\int_0^t \lambda_1(Lu_n(\xi) + R_1(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_1(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) d\xi - \int_0^t \lambda_1\big(g_1(\xi)\big) d\xi,$$

$\sum_{n=0}^{\infty} p^{(n)} v_n(x,t) = v_0(x,t) + p\int_0^t \lambda_2(Lv_n(\xi) + R_2(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_2(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n)d\xi - \int_0^t \lambda_2(g_2(\xi))d\xi,$

$\sum_{n=0}^{\infty} p^{(n)} w_n(x,t) = w_0(x,t) + p\int_0^t \lambda_3(Lw_n(\xi) + R_3(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) + N_3(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n)d\xi - \int_0^t \lambda_3(g_3(\xi))d\xi.$ (12)

A comparison of like powers of p gives various orders solutions.

Then, variational homotopy perturbation method is applied to homogeneous linear and nonlinear partial differential equation systems. The summary of the steps can be seen in Figure 1.

RESULTS AND DISCUSSION

In this section, variational homotopy perturbation method is applied to solve homogeneous linear and nonlinear partial differential equation system with two and three variables.

Variational Homotopy Perturbation on Homogeneous Linear Equation System

Consider homogeneous linear equation system as follow,

$$u_t - v_x + (u+v) = 0, \quad (13)$$

$$v_t - u_x + (u+v) = 0, \quad (14)$$

with the initial conditions

$$u(x,0) = \sinh x, \ v(x,0) = \cosh x \quad (15)$$

(Sami Bataineh et al., 2008). To solve system (13) – (15), we choose the initial approximations

$$u_0(x,0) = \sinh x, v_0(x,0) = \cosh x \quad (16)$$

Based on the variational iteration method, correct functionals of the system (13) – (14) are

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1 \left(Lu_n(\xi) + N_1(\tilde{u}_n, \tilde{v}_n) - g_1(\xi)\right) \delta\xi, \quad (17)$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1 \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial v_n(x,\xi)}{\partial x} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi)\right) \delta\xi; \quad (18)$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2 \left(Lv_n(\xi) + N_2(\tilde{u}_n, \tilde{v}_n) - g_2(\xi)\right) \delta\xi, \quad (19)$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2 \left(\frac{\partial v_n(x,\xi)}{\partial \xi} - \frac{\partial u_n(x,\xi)}{\partial x} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi)\right) \delta\xi; \quad (20)$$

The stationary conditions of the functionals (18) and (20) are given by

$1 + \lambda_1 = 0, \ \lambda_1'(\xi = t) = 0,$

$1 + \lambda_2 = 0, \ \lambda_2'(\xi = t) = 0.$

Consequently, we get $\lambda_1 = \lambda_2 = -1$ and then the functionals (18) and (20) give the following iteration schemes:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial v_n(x,\xi)}{\partial x} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi)\right) \delta\xi; \quad (21)$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left(\frac{\partial v_n(x,\xi)}{\partial \xi} - \frac{\partial u_n(x,\xi)}{\partial x} + \tilde{u}_n(x,\xi) + \tilde{v}_n(x,\xi)\right) \delta\xi; \quad (22)$$

where $n \geq 0$. Applying the homotopy perturbation method to (21) – (22), we obtain

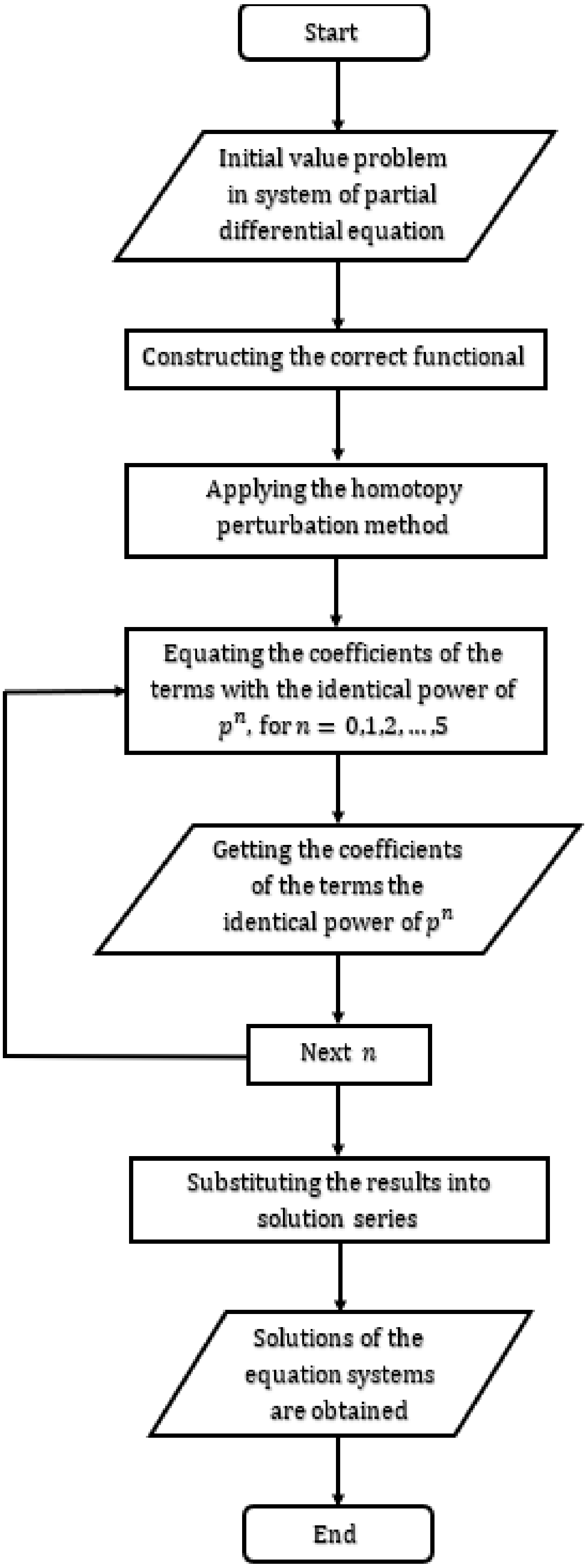

Figure 1. Algorithm of the variational homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n = u_0(x,t) - p\int_0^t \left(\left(\sum_{n=0}^{\infty} p^n u_n\right)_\xi - \left(\sum_{n=0}^{\infty} p^n v_n\right)_x + \left(\sum_{n=0}^{\infty} p^n u_n\right) + \left(\sum_{n=0}^{\infty} p^n v_n\right) \right) \delta\xi, \quad (23)$$

$$u_0 + pu_1 + p^2u_2 + \cdots = u_0(x,t) - p\int_0^t \left(\left(\frac{\partial u_0(x,\xi)}{\partial \xi} - \frac{\partial v_0(x,\xi)}{\partial x} + \tilde{u}_0(x,\xi) + \tilde{v}_0(x,\xi) \right) + p\left(\frac{\partial u_1(x,\xi)}{\partial \xi} - \frac{\partial v_1(x,\xi)}{\partial x} + \tilde{u}_1(x,\xi) + \tilde{v}_1(x,\xi) \right) + p^2\left(\frac{\partial u_2(x,\xi)}{\partial \xi} - \frac{\partial v_2(x,\xi)}{\partial x} + \tilde{u}_2(x,\xi) + \tilde{v}_2(x,\xi) \right) + \cdots \right) \delta\xi; \quad (24)$$

$$\sum_{n=0}^{\infty} p^n v_n = v_0(x,t) - p\int_0^t \left(\left(\sum_{n=0}^{\infty} p^n v_n\right)_\xi - \left(\sum_{n=0}^{\infty} p^n u_n\right)_x + \left(\sum_{n=0}^{\infty} p^n u_n\right) + \left(\sum_{n=0}^{\infty} p^n v_n\right) \right) \delta\xi, \quad (25)$$

$$v_0 + pv_1 + p^2v_2 + \cdots = v_0(x,t) - p\int_0^t \left(\left(\frac{\partial v_0(x,\xi)}{\partial \xi} - \frac{\partial u_0(x,\xi)}{\partial x} + \tilde{u}_0(x,\xi) + \tilde{v}_0(x,\xi) \right) + p\left(\frac{\partial v_1(x,\xi)}{\partial \xi} - \frac{\partial u_1(x,\xi)}{\partial x} + \tilde{u}_1(x,\xi) + \tilde{v}_1(x,\xi) \right) + p^2\left(\frac{\partial v_2(x,\xi)}{\partial \xi} - \frac{\partial u_2(x,\xi)}{\partial x} + \tilde{u}_2(x,\xi) + \tilde{v}_2(x,\xi) \right) + \cdots \right) \delta\xi. \quad (26)$$

Equating the coefficients of the terms with the identical powers of p, we get

p^0: $u_0(x,t) = \sinh x$; $v_0(x,t) = \cosh x$.

p^1: $u_1(x,t) = -t\cosh x$; $v_1(x,t) = -t\sinh x$.

p^2: $u_2(x,t) = \frac{t^2}{2!}\sinh x$; $v_2(x,t) = \frac{t^2}{2!}\cosh x$.

p^3: $u_3(x,t) = -\frac{t^3}{3!}\cosh x$; $v_3(x,t) = -\frac{t^3}{3!}\sinh x$.

p^4: $u_4(x,t) = \frac{t^4}{4!}\sinh x$; $v_4(x,t) = -\frac{t^4}{4!}\cosh x$.

p^5: $u_5(x,t) = -\frac{t^5}{5!}\cosh x$; $v_5(x,t) = -\frac{t^5}{5!}\sinh x$.

Then, the solution series of homotopy perturbation method are

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \quad (27)$$

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots \quad (28)$$

By setting $p = 1$, we find the solution of the equation system as follow

$$u(x,t) = \sinh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \cosh x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right); \quad (29)$$

$$v(x,t) = \cosh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \sinh x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right); \quad (30)$$

which will converge to $u(x,t) = \sinh(x - t)$ and $v(x,t) = \cosh(x - t)$. The solution we get is the same as with the solution obtained by homotopy analysis method (Sami Bataineh et al., 2008) and variational iteration method (Wazwaz, 2007).

Variational Homotopy Perturbation on Homogeneous Nonlinear Equation System

Given homogeneous nonlinear equation system,

$$u_t + u_xv_x + u_yv_y + u = 0, \quad (31)$$

$$v_t + v_xw_x - v_yw_y - v = 0, \quad (32)$$

$$w_t + w_xu_x + w_yu_y - w = 0, \quad (33)$$

with the initial conditions

$$u(x,y,0) = e^{x+y}, \quad v(x,y,0) = e^{x-y}, \quad w(x,y,0) = e^{-x+y} \tag{34}$$

(Sami Bataineh et al., 2008). To solve system (31) – (34), we choose the initial approximations

$$u_0(x,y,0) = e^{x+y}, \quad v_0(x,y,0) = e^{x-y}, \quad w_0(x,y,0) = e^{-x+y}. \tag{35}$$

By using the same procedure, construct correct functionals of system (31) – (33) as follow,

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda_1 \left(Lu_n(\xi) + N_1(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_1(\xi)\right) \delta\xi, \tag{36}$$

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda_1 \left(\frac{\partial u_n(x,y,\xi)}{\partial \xi} + \frac{\partial u_n(x,y,\xi)}{\partial x} . \frac{\partial v_n(x,y,\xi)}{\partial x} + \frac{\partial u_n(x,y,\xi)}{\partial y} . \frac{\partial v_n(x,y,\xi)}{\partial y} - \tilde{u}_n(x,y,\xi)\right) \delta\xi; \tag{37}$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) + \int_0^t \lambda_2 \left(Lv_n(\xi) + N_2(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_2(\xi)\right) \delta\xi, \tag{38}$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) + \int_0^t \lambda_2 \left(\frac{\partial v_n(x,y,\xi)}{\partial \xi} + \frac{\partial v_n(x,y,\xi)}{\partial x} . \frac{\partial w_n(x,y,\xi)}{\partial x} - \frac{\partial v_n(x,y,\xi)}{\partial y} . \frac{\partial w_n(x,y,\xi)}{\partial y} - \tilde{v}_n(x,y,\xi)\right) \delta\xi; \tag{39}$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) + \int_0^t \lambda_3 \left(Lv_n(\xi) + N_3(\tilde{u}_n, \tilde{v}_n, \widetilde{w}_n) - g_3(\xi)\right) \delta\xi, \tag{40}$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) + \int_0^t \lambda_3 \left(\frac{\partial w_n(x,y,\xi)}{\partial \xi} + \frac{\partial w_n(x,y,\xi)}{\partial x} . \frac{\partial u_n(x,y,\xi)}{\partial x} + \frac{\partial w_n(x,y,\xi)}{\partial y} . \frac{\partial u_n(x,y,\xi)}{\partial y} - \widetilde{w}_n(x,y,\xi)\right) \delta\xi. \tag{41}$$

This yields the stationary conditions

$1 + \lambda_1 = 0,\ \lambda_1'(\xi = t) = 0,$
$1 + \lambda_2 = 0,\ \lambda_2'(\xi = t) = 0.$
$1 + \lambda_3 = 0,\ \lambda_3'(\xi = t) = 0.$

As a result, we find $\lambda_1 = \lambda_2 = \lambda_3 = -1$ such that the functionals (36), (38) and (40) give the following iteration formulas.

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t \left(\frac{\partial u_n(x,y,\xi)}{\partial \xi} + \frac{\partial u_n(x,y,\xi)}{\partial x} . \frac{\partial v_n(x,y,\xi)}{\partial x} + \frac{\partial u_n(x,y,\xi)}{\partial y} . \frac{\partial v_n(x,y,\xi)}{\partial y} - \tilde{u}_n(x,y,\xi)\right) \delta\xi; \tag{42}$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) - \int_0^t \left(\frac{\partial v_n(x,y,\xi)}{\partial \xi} + \frac{\partial v_n(x,y,\xi)}{\partial x} . \frac{\partial w_n(x,y,\xi)}{\partial x} - \frac{\partial v_n(x,y,\xi)}{\partial y} . \frac{\partial w_n(x,y,\xi)}{\partial y} - \tilde{v}_n(x,y,\xi)\right) \delta\xi; \tag{43}$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) - \int_0^t \left(\frac{\partial w_n(x,y,\xi)}{\partial \xi} + \frac{\partial w_n(x,y,\xi)}{\partial x} . \frac{\partial u_n(x,y,\xi)}{\partial x} + \frac{\partial w_n(x,y,\xi)}{\partial y} . \frac{\partial u_n(x,y,\xi)}{\partial y} - \widetilde{w}_n(x,y,\xi)\right) \delta\xi; \tag{44}$$

where $n \geq 0$. Applying the homotopy perturbation method to (42) – (44), we have

$$\sum_{n=0}^{\infty} p^n u_n = u_0(x,y,t) - p \int_0^t \Big(\left(\sum_{n=0}^{\infty} p^n u_n\right)_\xi + \left(\sum_{n=0}^{\infty} p^n u_n\right)_x \left(\sum_{n=0}^{\infty} p^n v_n\right)_x + \left(\sum_{n=0}^{\infty} p^n u_n\right)_y \left(\sum_{n=0}^{\infty} p^n v_n\right)_y + \left(\sum_{n=0}^{\infty} p^n u_n\right) \Big) \delta\xi \tag{45}$$

$$u_0 + pu_1 + p^2 u_2 + \cdots = u_0(x,y,t) - p \int_0^t \Bigg(\left(\frac{\partial u_0(x,y,\xi)}{\partial \xi} + \frac{\partial u_0(x,y,\xi)}{\partial x} . \frac{\partial v_0(x,y,\xi)}{\partial x} + \frac{\partial u_0(x,y,\xi)}{\partial y} . \frac{\partial v_0(x,y,\xi)}{\partial y} + \tilde{u}_0(x,y,\xi)\right) + p\left(\frac{\partial u_1(x,y,\xi)}{\partial \xi} + \frac{\partial u_1(x,y,\xi)}{\partial x} . \frac{\partial v_1(x,y,\xi)}{\partial x} + \frac{\partial u_1(x,y,\xi)}{\partial y} . \frac{\partial v_1(x,y,\xi)}{\partial y} + \tilde{u}_1(x,y,\xi)\right) +$$

$$p^2\left(\frac{\partial u_2(x,y,\xi)}{\partial \xi} + \frac{\partial u_2(x,y,\xi)}{\partial x}.\frac{\partial v_2(x,y,\xi)}{\partial x} + \frac{\partial u_2(x,y,\xi)}{\partial y}.\frac{\partial v_2(x,y,\xi)}{\partial y} + \tilde{u}_2(x,y,\xi)\right) + \cdots\Bigg)\delta\xi; \quad (46)$$

$$\sum_{n=0}^{\infty} p^n v_n = v_0(x,y,t) - p\int_0^t \left(\left(\sum_{n=0}^{\infty} p^n v_n\right)_\xi + \left(\sum_{n=0}^{\infty} p^n v_n\right)_x \left(\sum_{n=0}^{\infty} p^n w_n\right)_x - \left(\sum_{n=0}^{\infty} p^n v_n\right)_y \left(\sum_{n=0}^{\infty} p^n w_n\right)_y - \left(\sum_{n=0}^{\infty} p^n v_n\right)\right)\delta\xi \quad (47)$$

$$v_0 + pv_1 + p^2v_2 + \cdots = v_0(x,y,t) - p\int_0^t \Bigg(\left(\frac{\partial v_0(x,y,\xi)}{\partial \xi} + \frac{\partial v_0(x,y,\xi)}{\partial x}.\frac{\partial w_0(x,y,\xi)}{\partial x} - \frac{\partial v_0(x,y,\xi)}{\partial y}.\frac{\partial w_0(x,y,\xi)}{\partial y} - \tilde{v}_0(x,y,\xi)\right) + p\left(\frac{\partial v_1(x,y,\xi)}{\partial \xi} + \frac{\partial v_1(x,y,\xi)}{\partial x}.\frac{\partial w_1(x,y,\xi)}{\partial x} - \frac{\partial v_1(x,y,\xi)}{\partial y}.\frac{\partial w_1(x,y,\xi)}{\partial y} - \tilde{v}_1(x,y,\xi)\right) + p^2\left(\frac{\partial v_2(x,y,\xi)}{\partial \xi} + \frac{\partial v_2(x,y,\xi)}{\partial x}.\frac{\partial w_2(x,y,\xi)}{\partial x} - \frac{\partial v_2(x,y,\xi)}{\partial y}.\frac{\partial w_2(x,y,\xi)}{\partial y} - \tilde{v}_2(x,y,\xi)\right) + \cdots\Bigg)\delta\xi \quad (48)$$

$$\sum_{n=0}^{\infty} p^n w_n = w_0(x,y,t) - p\int_0^t \left(\left(\sum_{n=0}^{\infty} p^n w_n\right)_\xi + \left(\sum_{n=0}^{\infty} p^n w_n\right)_x \left(\sum_{n=0}^{\infty} p^n u_n\right)_x + \left(\sum_{n=0}^{\infty} p^n w_n\right)_y \left(\sum_{n=0}^{\infty} p^n u_n\right)_y - \left(\sum_{n=0}^{\infty} p^n w_n\right)\right)\delta\xi \quad (49)$$

$$w_0 + pw_1 + p^2w_2 + \cdots = w_0(x,y,t) - p\int_0^t \Bigg(\left(\frac{\partial w_0(x,y,\xi)}{\partial \xi} + \frac{\partial w_0(x,y,\xi)}{\partial x}.\frac{\partial u_0(x,y,\xi)}{\partial x} - \frac{\partial w_0(x,y,\xi)}{\partial y}.\frac{\partial u_0(x,y,\xi)}{\partial y} - \widetilde{w}_0(x,y,\xi)\right) + p\left(\frac{\partial w_1(x,y,\xi)}{\partial \xi} + \frac{\partial w_1(x,y,\xi)}{\partial x}.\frac{\partial u_1(x,y,\xi)}{\partial x} - \frac{\partial w_1(x,y,\xi)}{\partial y}.\frac{\partial u_1(x,y,\xi)}{\partial y} - \widetilde{w}_1(x,y,\xi)\right) + p^2\left(\frac{\partial w_2(x,y,\xi)}{\partial \xi} + \frac{\partial w_2(x,y,\xi)}{\partial x}.\frac{\partial u_2(x,y,\xi)}{\partial x} - \frac{\partial w_2(x,y,\xi)}{\partial y}.\frac{\partial u_2(x,y,\xi)}{\partial y} - \widetilde{w}_2(x,y,\xi)\right) + \cdots\Bigg)\delta\xi \quad (50)$$

From Equation (46), (48) and (50) we get

p^0: $u_0(x,y,t) = e^{x+y}$; $v_0(x,y,t) = e^{x-y}$; $w_0(x,y,t) = e^{-x+y}$.

p^1: $u_1(x,y,t) = -\,te^{x+y}$; $v_1(x,y,t) = te^{x-y}$; $w_1(x,y,t) = te^{-x+y}$.

p^2: $u_2(x,y,t) = \frac{t^2e^{x+y}}{2!}$; $v_2(x,y,t) = \frac{t^2e^{x-y}}{2!}$; $w_2(x,y,t) = \frac{t^2e^{-x+y}}{2!}$.

p^3: $u_3(x,y,t) = -\frac{t^3e^{x+y}}{3!}$; $v_3(x,y,t) = \frac{t^3e^{x-y}}{3!}$; $w_3(x,y,t) = \frac{t^3e^{-x+y}}{3!}$.

p^4: $u_4(x,y,t) = \frac{t^4e^{x+y}}{4!}$; $v_4(x,y,t) = \frac{t^4e^{x-y}}{4!}$; $w_4(x,y,t) = \frac{t^4e^{-x+y}}{4!}$.

p^5: $u_5(x,y,t) = -\frac{t^5e^{x+y}}{5!}$; $v_5(x,y,t) = \frac{t^5e^{x-y}}{5!}$; $w_5(x,y,t) = \frac{t^5e^{-x+y}}{5!}$.

Then, the solution series of homotopy perturbation method is given by

$$u(x,t) = \sum_{n=0}^{\infty} p^n\, u_n(x,t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots; \quad (51)$$

$$v(x,t) = \sum_{n=0}^{\infty} p^n\, v_n(x,t) = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots \quad (52)$$

$$w(x,y,t) = \sum_{n=0}^{\infty} p^n\, w_n(x,y,t) = w_0 + pw_1 + p^2w_2 + p^3w_3 + \cdots \quad (53)$$

Like in the previous problem, set $p = 1$ to get the solution of system (31)-(34) as follow

$$u(x,y,t) = e^{x+y}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\right); \quad (54)$$

$$v(x,y,t) = e^{x-y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right); \quad (55)$$

$$w(x,y,t) = e^{-x+y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right); \quad (56)$$

which will converge to $u(x,y,t) = e^{x+y-t}$; $v(x,y,t) = e^{x-y+t}$ and $w(x,y,t) = e^{-x+y+t}$. Once again the solution we get is the same as the solution obtained by homotopy analysis method (Sami Bataineh et al., 2008) and variational iteration method (Wazwaz, 2007).

Based on the previous calculations, this method is proven to be capable of reducing the computational work compared to homotopy analysis method and variational iteration method while still maintaining the high accuracy of the result.

CONCLUSIONS AND SUGGESTIONS

In this research, variational homotopy perturbation on homogeneous linear and nonlinear partial differential equation system with two and three variables have been applied. Based on the results obtained, the solutions matched the solution obtained by homotopy analysis method and variational iteration method, which proves the validity of the variational homotopy perturbation method when applied to system of partial differential equations. From the previous sections, it can be concluded that variational homotopy perturbation method gives the solution by using the initial conditions only. Besides that, this method is capable of reducing the computational work while still maintaining the high accuracy of the result.

In this research, the variational homotopy perturbation method was applied to the systems of selected partial differential equations only, specifically, homogeneous linear and nonlinear partial differential equation system with two and three variables. Therefore, it is hoped that it can be used as a reference in applying the variational method of perturbation homotopy to other more complicated systems of partial differential equations.

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