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Odd harmonious labeling on the union of flower graphs

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ABSTRACT

Applications of graph labeling in the fields of communication network addressing, database management, secret sharing schemes, and cryptology. Graphs that satisfy the odd harmonious labeling property are called odd harmonious graphs. The purposes of the research are to obtain the construction of the union of zinnia flower graphs, the union of double quadrilateral flower graphs, the rosella flower graphs, and the union of rosella flower graphs. The research method consists of literature study, graph class construction, graph labeling construction, theorem construction, and proof. The result of the research proves that the union of the zinnia flower graph, the double quadrilateral flower graph, the rosella flower graph, and the union of the rosella flower graph satisfies the odd harmonious labeling property. Thus, the novelty of this research is that the properties of the new graph class of odd harmonious graphs are obtained.

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INTRODUCTION

Graph labeling is one of the most rapidly growing research topics in graph theory in recent years. Researchers have discovered several types of graph labeling and their properties, including magic labeling, anti-magic labeling, graceful labeling, harmonious labeling, odd harmonious labeling, and even harmonious labeling. Graph labeling is basically labeling vertices and edges with certain rules and patterns.

Gallian (2022) has collected research papers on graph labeling and its applications. Odd harmonious labeling was introduced by Liang & Bai (2009). A graph $G(p,q)$ with $p=|V(G)|$ and $q=|E(G)|$ is an odd harmonious graph if it satisfies an injective vertex labeling function $f: V(G) \to \{0,1,2,3,4,\dots,2q-1\}$ inducing a bijective edge labeling function $f^*: V(G) \to \{0,1,2,3,4,\dots,2q-1\}$ (Liang & Bai, 2009).

The following are some graph classes that have been successfully discovered and are odd harmonious graph families. Abdel-Aal (2013) proved that cyclic snake graphs are odd harmonic graphs. Saputri, Sugeng, & Froncek (2013) proved that dumbbell graphs and the generalization of prism graphs are odd harmonious graphs.

Jeyanthi & Philo (2015) proved that some classes of double quadrilateral snake graphs are odd harmonious graphs. In another paper, Jeyanthi & Philo (2016) proved that some cycle-related graphs are odd harmonious graphs. Abdel-Aal & Seoud (2016) proved that m-shadow graphs for paths and complete bipartite graphs are odd harmonious graphs.

Firmansah & Yuwono (2017a) proved that the pleated Dutch windmill graph is an odd harmonious graph; in a different paper, Firmansah & Yuwono (2017b) proved that the class of graphs resulting from the Cartesian product operation is an odd harmonious graph. Firmansah (2017) proved that the variation of the double quadrilateral windmill graph is an odd harmonious graph. Seoud & Hafez (2018) introduced the strong odd harmonious graph.

Sugeng, Surip, & Rismayati (2019) introduced m-shadows of cycle graphs, gear graphs with pendants, and shuriken graphs, which are odd harmonious graphs. Jeyanthi, Philo, & Youssef (2019) proved that grid graphs are odd harmonious graphs. In addition, Jeyanthi & Philo (2019) proved that the subdivided shell graph is an odd harmonious graph. Febriana & Sugeng (2020) proved that squid graphs and double squid graphs are odd harmonious graphs.

Firmansah & Giyarti (2021) proved that the amalgamation of the generalized double quadrilateral windmill graph is an odd harmonious graph. Pujiwati, Halikin, & Wijaya (2021) proved that two-star graphs are odd harmonious graphs. Sarasvati, Halikin, & Wijaya (2021) proved that PnC4 and PnD2(C4) graphs are odd harmonious graphs. Philo & Jeyanthi (2021) proved that line and disjoint union of graphs are odd harmonious graphs.

Firmansah (2022) proved that some sting graphs are odd harmonious graphs. Firmansah (2023) proved that layered graphs are odd harmonious graphs. Lasim, Halikin, & Wijaya (2022) proved that there is a relationship between harmonic graphs, odd harmonious graphs, and even harmonious graphs. Hafez, El-Shanawany, & Atik (2023) proved that the converse skew product of graphs is an odd harmonious graph. Kolo, Ginting, & Putra (2023) proved that the graph Cm,nC4 is an odd harmonious graph.

Firmansah (2020) proved that the double quadrilateral flower graph is an odd harmonious graph. In another paper, Firmansah, Tasari, & Yuwono (2023) proved that the zinnia flower graph and its variations are odd harmonious graphs.

These two research results are the basis for researchers to develop new graphs in the form of a union of double quadrilateral flower graphs and zinnia flower graphs. In addition, the author will introduce new graphs, namely the rosella flower graphs and the union of rosella flower graphs. So the novelty of this research is that the definition of the union of zinnia flower graphs, the union of double quadrilateral flower graphs, rosella flower graphs, and the union of the rosella flower graphs are obtained. Furthermore, it will be proven that the union of zinnia flower graphs, the union of double quadrilateral flower graphs, rosella flower graphs, and the union of the rosella flower graphs are odd harmonious graphs.

METHOD

This research is qualitative research with the aim of obtaining a new graph class of odd harmonious graphs so that the properties of odd harmonious graphs are obtained.

The research stages consist of 1) the literature study stage, which is to collect unsolved open problems from previous researchers, 2) the graph construction stage, which is to create a new graph with vertex and edge notation, 3) the label construction stage, which is to label the vertices with a certain pattern to get the vertex labeling function, then an edge labeling function will be formed, which is obtained from the induction of the vertex labeling function, 4) the result verification stage, which is the formation of a theorem to prove that the vertex labeling function is injective and induces a bijective edge labeling function.

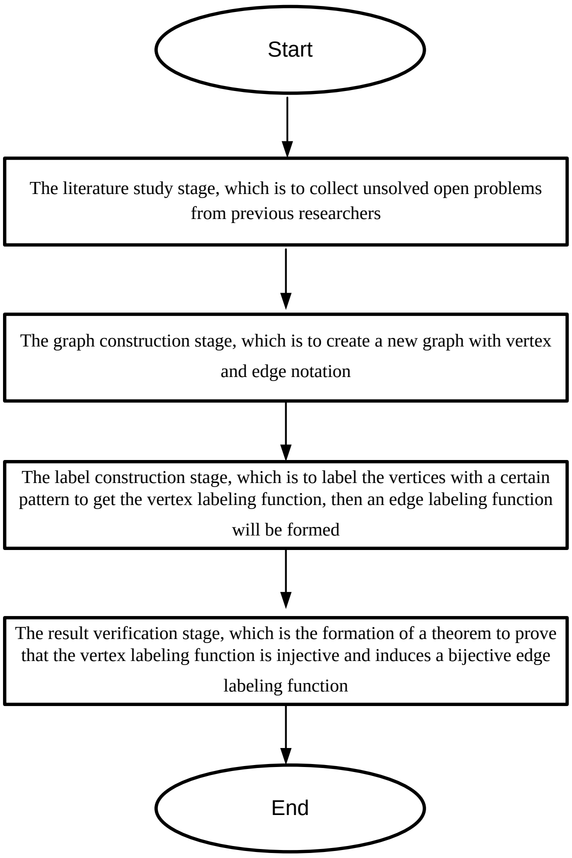

Figure 1. Flowchart research method

RESULTS AND DISCUSSION

Two new graph classes that are extensions of the zinnia flower graphs and double quadrilateral flower graphs are presented in this chapter. A graph created by the union operations of two zinnia flower graphs is known as the union of zinnia flower graphs. A graph created by the union operation of two double quadrilateral flower graphs is known as the union of double quadrilateral flower graphs.

The definition and construction of the union of zinnia flower graphs are provided in this section. It will also be demonstrated that the union of zinnia flower graphs is an odd harmonious graph.

Definition 1. The union of zinnia flower graphs $Z(h) \cup Z(h)$ with $h \geq 1$ as graph that has vertex set
$V\big(Z(h) \cup Z(h)\big) = \{a_j \mid 1 \leq j \leq 2h+2\} \cup \{b_i \mid i = 1,2\} \cup \{{c_j}^i \mid 1 \leq j \leq h, i = 1,2\} \cup \{w_j \mid j = 1,2\} \cup \{x_i \mid i = 1,2\} \cup \{y_j \mid 1 \leq j \leq 2h\} \cup \{{z_j}^i \mid 1 \leq j \leq h, i = 1,2\}$
and edge set
$E\big(Z(h) \cup Z(h)\big) =$
$\{a_j b_i \mid 1 \leq j \leq 2h+2, i = 1,2\} \cup$
$\{a_1 {c_j}^i \mid 1 \leq j \leq h\} \cup$
$\{a_2 {c_j}^i \mid 1 \leq j \leq h, i = 1,2\} \cup$
$\{w_j x_i \mid i,j = 1,2\} \cup$
$\{x_i y_j \mid 1 \leq j \leq 2h, i = 1,2\} \cup$
$\{w_1 {z_j}^i \mid 1 \leq j \leq h, i = 1,2\} \cup$
$\{w_2 {z_j}^i \mid 1 \leq j \leq h, i = 1,2\}$.

Based on Definition 1, we obtained that order $p = \big|V\big(Z(h) \cup Z(h)\big)\big| = 8h+8$ and size $q = \big|E\big(Z(h) \cup Z(h)\big)\big| = 16h+8$. The following is the construction of the union of zinnia flower graphs in Figure 2.

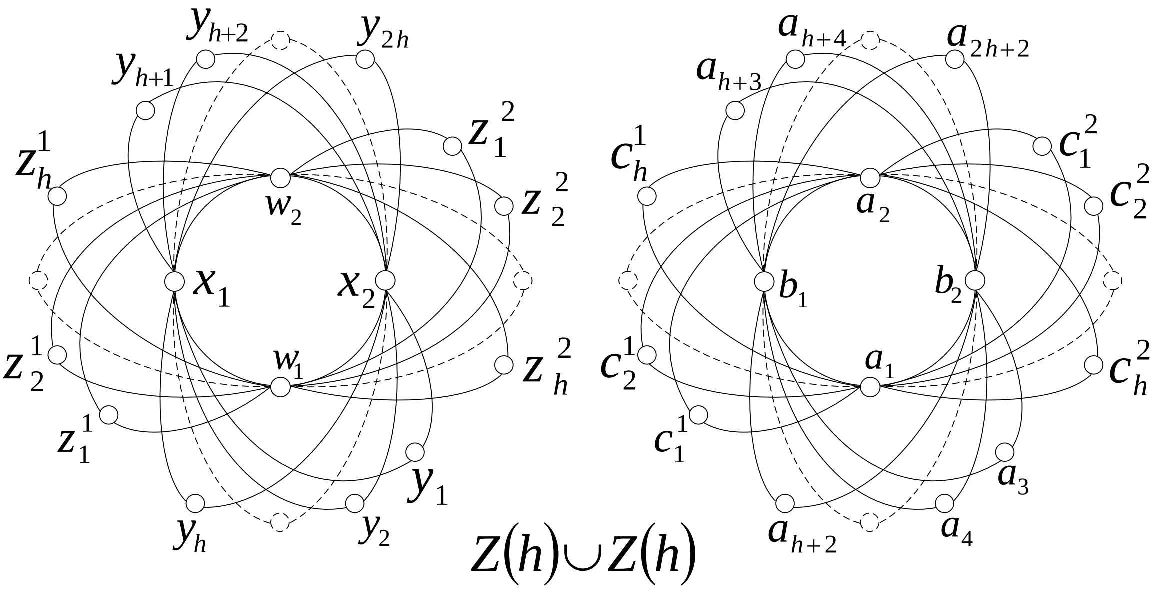

Figure 2. Construction of the Union of Zinnia Flower Graphs $Z(h) \cup Z(h)$

Theorem 1. The union of zinnia flower graphs $Z(h) \cup Z(h)$ with $h \geq 1$ are odd harmonious graphs

Proof.

Define the vertex labeling function of graphs $Z(h) \cup Z(h)$ with $h \geq 1$ as follows

$f(a_j) = 4j - 2, 1 \leq j \leq 2h + 2$ (1)

$f(b_i) = 2i + 5, i = 1,2$ (2)

$f(c_j{}^i) = 8h + 8j + 2i + 5,$
$1 \leq j \leq h, i = 1,2$ (3)

$f(w_j) = 4j - 4, j = 1,2$ (4)

$f(x_i) = 2i - 1, i = 1,2$ (5)

$f(y_j) = 16h + 4j + 12, 1 \leq j \leq 2h$ (6)

$f(z_j{}^i) = 24h + 8j + 2i + 7,$
$1 \leq j \leq h, i = 1,2$ (7)

Based on (1), (2), (3), (4), (5), (6), and (7)

$f\left(V\big(Z(h) \cup Z(h)\big)\right) = \{2,6,10,14,\dots,8h - 6\} \cup \{7,9\} \cup \{8h + 15, 8h + 23, 8h + 31,\dots,16h + 7, 8h + 17, 8h + 25, 8h + 33,\dots,16h + 9\} \cup \{0,4\} \cup \{1,3\} \cup \{16h + 16, 16h + 20, 16h + 24,\dots,24h + 12\} \cup \{24h + 17, 24h + 25, 24h + 33,\dots,32h + 9, 24h + 19, 24h + 27, 24h + 35,\dots,32h + 11\}$
$= \{2,6,7,9,10,14,\dots,8h - 6, 8h + 15, 8h + 17, 8h + 23, 8h + 25, 8h + 31, 8h + 33,\dots,16h + 7, 16h + 9\} \cup \{0,1,3,4,16h + 16, 16h + 20, 16h + 24,\dots,24h + 12, 24h + 17, 24h + 19, 24h + 25, 24h + 27, 24h + 33, 24h + 35,\dots,32h + 9, 32h + 11\}$
$= \{0,1,2,3,4,6,7,9,10,14,\dots,8h - 6, 8h + 15,\dots,16h + 7, 16h + 9, 16h + 16, 16h + 20,\dots,24h + 12, 24h + 17,\dots,32h + 9, 32h + 11\}$
$= \{0,1,2,3,4,6,\dots,32h + 11\}$
$\subseteq \{0,1,2,3,4,\dots,32h + 15\}$

and different labels on each vertex, then the vertex labeling function $f: V\big(Z(h) \cup Z(h)\big) \to \{0,1,2,3,4,\dots,32h + 15\}$ is injective.

Define the edge labeling function of graphs $Z(h) \cup Z(h)$ with $h \geq 1$ as follows

$f^*(a_j b_i) = 4j + 2i + 3,$
$1 \leq j \leq 2h + 2, i = 1,2$ (8)

$f^*(a_1 c_j{}^i) = 8h + 8j + 2i + 7,$
$1 \leq j \leq h, i = 1,2$ (9)

$f^*(a_2 c_j{}^i) = 8h + 8j + 2i + 11,$
$1 \leq j \leq h, i = 1,2$ (10)

$f^*(w_j x_i) = 4j + 2i - 5, i,j = 1,2$ (11)

$f^*(x_i y_j) = 16h + 4j + 2i + 11,$
$1 \leq j \leq 2h, i = 1,2$ (12)

$f^*(w_1 z_j{}^i) = 24h + 8j + 2i + 7,$
$1 \leq j \leq h, i = 1,2$ (13)
$f^*(w_2 z_j{}^i) = 24h + 8j + 2i + 11,$
$1 \leq j \leq h, i = 1,2$ (14)

Based on (8), (9), (10), (11), (12), (13), and (14)
$f^*\left(E\big(Z(h) \cup Z(h)\big)\right) = \{9,13,17, \dots ,8h + 13,11,15,19, \dots ,8h + 15\} \cup \{8h + 17,8h + 25,8h + 33, \dots ,16h + 9,8h + 19,8h + 27,8h + 35, \dots ,16h + 11\} \cup \{8h + 21,8h + 29,8h + 37, \dots ,16h + 13, 8h + 23,8h + 31,8h + 39, \dots ,16h + 15\} \cup \{1,3,5,7\} \cup \{16h + 17,16h + 21,16h + 25, \dots ,24h + 13,16h + 19,16h + 23,16h + 27, \dots ,24h + 15\} \cup \{24h + 17,24h + 25,24h + 33, \dots ,32h + 9,24h + 19,24h + 27,24h + 35, \dots ,32h + 11\} \cup \{24h + 21,24h + 29,24h + 37, \dots ,32h + 13,24h + 23,24h + 31,24h + 39, \dots ,32h + 15\}$
$= \{9,11,13,15,17,19, \dots ,8h + 13,8h + 15,8h + 17,8h + 19,8h + 21,8h + 23,8h + 25,8h + 27,8h + 29,8h + 31,8h + 35,8h + 37,8h + 39, \dots ,16h + 9,16h + 11,16h + 13,16h + 15\} \cup \{1,3,5,7,16h + 17,16h + 19,16h + 21,16h + 23,16h + 25,16h + 27, \dots ,24h + 13,24h + 15,24h + 17,24h + 19,24h + 21,24h + 23,24h + 25,24h + 27,24h + 29,24h + 31,24h + 33, \dots ,32h + 9,32h + 11,32h + 13,32h + 15\}$
$= \{1,3,5,7,9,11,13,15,17,19, \dots ,8h + 13,8h + 15,8h + 17,8h + 19,8h + 21,8h + 23,8h + 25,8h + 27,8h + 29,8h + 31,8h + 35,8h + 37,8h + 39, \dots ,16h + 9,16h + 11,16h + 13,16h + 15,16h + 17,16h + 19,16h + 21,16h + 23,16h + 25,16h + 27, \dots ,24h + 13,24h + 15,24h + 17,24h + 19,24h + 21,24h + 23,24h + 25,24h + 27,24h + 29,24h + 31,24h + 33, \dots ,32h + 9,32h + 11,32h + 13,32h + 15\}$
$= \{1, 3, 5, 7, \dots ,32h + 15\}$
and different labels on each edge, then the edge labeling function $f^*: E\big(Z(h) \cup Z(h)\big) \to \{1, 3, 5, 7, \dots ,32h + 15\}$ is bijective.

Such that the union of zinnia flower graphs $Z(h) \cup Z(h)$ with $h \geq 1$ are odd harmonious graphs ■

The following two examples are given for graph $Z(3) \cup Z(3)$ in Figure 3 and graph $Z(4) \cup Z(4)$ in Figure 4, which are odd harmonious graphs.

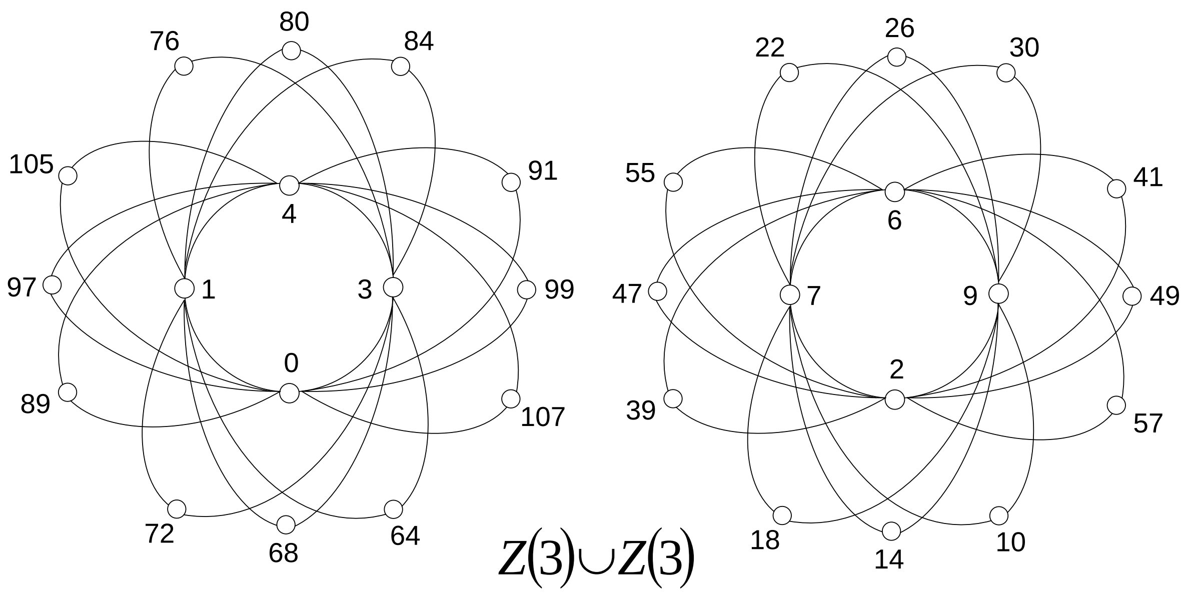

Figure 3. Odd Harmonious Labeling on the Graph $Z(3) \cup Z(3)$

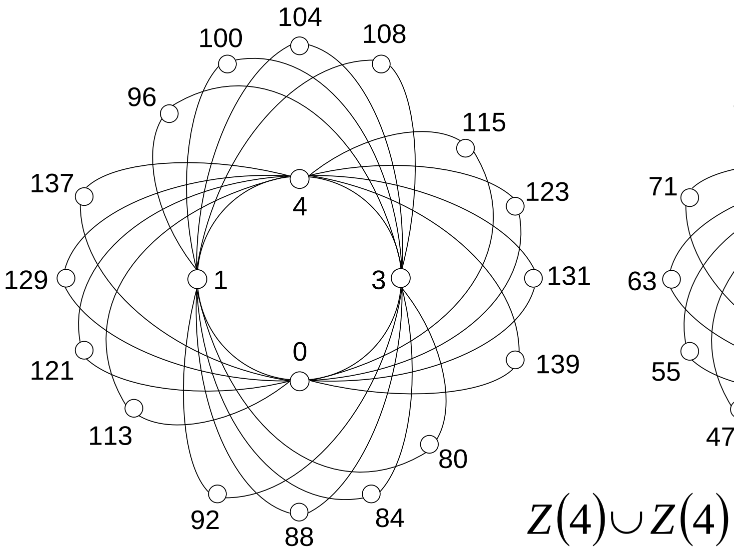

Figure 4. Odd Harmonious Labeling on the Graph $Z(4) \cup Z(4)$

The definition and construction of the union of double quadrilateral flower graphs are provided in this section. It will also be demonstrated that the union of double quadrilateral flower graphs are odd harmonious graphs.

Definition 2. The union of double qudrilateral flower graphs $D(r) \cup D(r)$ with $r \geq 1$ as graph that has vertex set $V\big(D(r) \cup D(r)\big) = \{k_0\} \cup$
$\{k_j \mid 1 \leq j \leq 2r+1\} \cup \{l_j \mid 1 \leq j \leq 2r\} \cup$
$\{m_0\} \cup \{m_j \mid 1 \leq j \leq 2r+1\} \cup$
$\{n_j \mid 1 \leq j \leq 2r\}$
and edge set

$E\big(D(r) \cup D(r)\big) =$
$\{k_0 k_j \mid 1 \leq j \leq 2r+1\} \cup$
$\{k_{2j-1} l_{2j-1} \mid 1 \leq j \leq r\} \cup$
$\{k_{2j+1} l_{2j} \mid 1 \leq j \leq r\} \cup$
$\{l_{2j-1} k_{2j} \mid 1 \leq j \leq r\} \cup$
$\{l_{2j} k_{2j} \mid 1 \leq j \leq r\} \cup$
$\{m_0 m_j \mid 1 \leq j \leq 2r+1\} \cup$
$\{m_{2j-1} n_{2j-1} \mid 1 \leq j \leq r\} \cup$
$\{m_{2j+1} n_{2j} \mid 1 \leq j \leq r\} \cup$
$\{n_{2j-1} m_{2j} \mid 1 \leq j \leq r\} \cup$
$\{n_{2j} m_{2j} \mid 1 \leq j \leq r\}$

Based on Definition 3, we obtained that order $p = \big|V\big(D(r) \cup D(r)\big)\big| = 8r+4$ and size $q = \big|E\big(D(r) \cup D(r)\big)\big| = 12r+2$. The following is given the construction of the union double quadrilateral flower graphs in Figure 5.

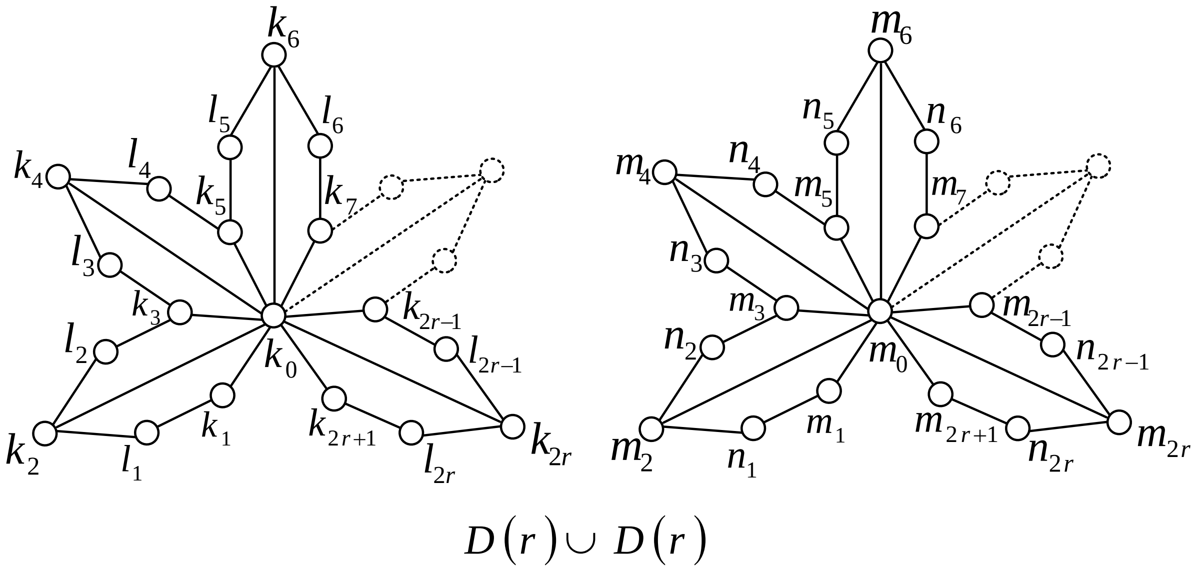

Figure 5. Construction of the Union of Double Quadrilateral Flower Graph $D(r) \cup \mathrm{D}(r)$

Theorem 2. The union of double quadrilateral flower graphs $D(r) \cup D(r)$ with $r \geq 1$ are odd harmonious graphs

Proof.

Define the vertex labeling function of graphs $D(r) \cup D(r)$ with $r \geq 1$ as follows

$f(k_0) = 0$ (15)

$f(k_j) = 2j - 1, 1 \leq j \leq 2r + 1$ (16)

$f(l_j) = 12r - 6j + 4, 1 \leq j \leq 2r$ (17)

$f(m_0) = 2$ (18)

$f(m_j) = 12r + 2j - 1,$
$1 \leq j \leq 2r + 1$ (19)

$f(n_j) = 12r - 6j + 6, 1 \leq j \leq 2r$ (20)

Based on (15), (16), (17), (18), (19), and (20)

$f\left(V\left(D(r) \cup D(r)\right)\right) = \{0\} \cup \{1,3,5,7,9, \dots, 4r+1\} \cup \{12r-2, 12r-8, 12r-14, 12r-20, \dots, 22,16,10,4\} \cup \{2\} \cup \{12r+1, 12r+3, 12r+5, 12r+7, \dots, 16r+1\} \cup \{12r, 12r-6, 12r-12, 12r-18, \dots, 24,18,12,6\}$
$= \{0,1,3,4,5,7,9,10, \dots, 4r+1, \dots, 12r-20, 12r-14, 12r-8, 12r-2\} \cup \{2,6,12,18,24, \dots, 12r-18, 12r-12, 12r-6, 12r, 12r+1, 12r+3, 12r+5, 12r+7, \dots, 16r+1\}$
$= \{0,1,2,3,4,5,6,7,9,10,12, \dots, 4r+1, .., 12r-20, 12r-18, 12r-14, 12r-12, 12r-8, 12r-6, 12r-2, 12r, 12r+1, 12r+3, 12r+5, 12r+7, \dots, 16r+1\}$
$= \{0,1,2,3,4, \dots, 16r+1\}$
$\subseteq \{0,1,2,3,4, \dots, 24r+3\}$

obtained different labels on each vertex, and the vertex labeling function $f: V\left(D(r) \cup D(r)\right) \rightarrow \{0,1,2,3,4, \dots, 24r+3\}$ is injective.

Define the edge labeling function of graphs $D(r) \cup D(r)$ with $r \geq 1$ as follows

$f^*(k_0 k_j) = 2j - 1, 1 \leq j \leq 2r + 1$ (21)

$f^*(k_{2j-1} l_{2j-1}) = 12r - 8j + 7,$
$1 \leq j \leq r$ (22)

$f^*(k_{2j+1} l_{2j}) = 12r - 8j + 5,$
$1 \leq j \leq r$ (23)

$f^*(l_{2j-1} k_{2j}) = 12r - 8j + 9,$
$1 \leq j \leq r$ (24)

$f^*(l_{2j} k_{2j}) = 12r - 8j + 3,$
$1 \leq j \leq r$ (25)

$f^*(m_0 m_j) = 12r + 2j + 1,$
$1 \leq j \leq 2r + 1$ (26)

$f^*(m_{2j-1} n_{2j-1}) = 24r - 8j + 9,$
$1 \leq j \leq r$ (27)

$f^*(m_{2j+1} n_{2j}) = 24r - 8j + 7,$
$1 \leq j \leq r$ (28)

$f^*(n_{2j-1} m_{2j}) = 24r - 8j + 11,$
$1 \leq j \leq r$ (29)

$f^*(n_{2j} m_{2j}) = 24r - 8j + 5,$

$$1 \leq j \leq r \qquad (30)$$

Based on (21), (22), (23), (24), (25), (26), (27), (28), (29), and (30)

$f^*\left(E\big(D(r) \cup D(r)\big)\right) = \{1,3,5,7,9, \dots, 4r + 1\} \cup \{12r - 1, 12r - 9, 12r - 17, 12r - 25, \dots, 4r + 7\} \cup \{12r - 3, 12r - 11, 12r - 19, 12r - 27, \dots, 4r + 5\} \cup \{12r + 1, 12r - 7, 12r - 15, 12r - 23, \dots, 4r + 9\} \cup \{12r - 5, 12r - 13, 12r - 21, 12r - 29, \dots, 4r + 3\} \cup \{12r + 3, 12r + 5, 12r + 7, 12r + 9, \dots, 16r + 3\} \cup \{24r + 1, 24r - 7, 24r - 15, 24r - 23, \dots, 16r + 9\} \cup \{24r - 1, 24r - 9, 24r - 17, 24r - 25, \dots, 16r + 7\} \cup \{24r + 3, 24r - 5, 24r - 13, 24r - 21, \dots, 16r + 11\} \cup \{24r - 3, 24r - 11, 24r - 19, 24r - 27, \dots, 16r + 5\}$
$= \{1,3,5,7,9, \dots, 4r + 1, 4r + 3, 4r + 5, 4r + 7, 4r + 9, \dots, 2r - 29, 12r - 27, 12r - 25, 12r - 23, 12r - 21, 12r - 19, 12r - 17, 12r - 15, 12r - 13, 12r - 11, 12r - 9, 12r - 7, 12r - 5, 12r - 3, 12r - 1, 12r + 1\} \cup \{12r + 3, 12r + 5, 12r + 7, 12r + 9, \dots, 16r + 3, 16r + 5, 16r + 7, 16r + 9, 16r + 11, \dots, 24r - 27, 24r - 25, 24r - 23, 24r - 21, 24r - 19, 24r - 17, 24r - 15, 24r - 13, 24r - 11, 24r - 9, 24r - 7, 24r - 5, 24r - 3, 24r - 1, 24r + 1, 24r + 3\}$
$= \{1,3,5,7,9, \dots, 4r + 1, 4r + 3, 4r + 5, 4r + 7, 4r + 9, \dots, 2r - 29, 12r - 27, 12r - 25, 12r - 23, 12r - 21, 12r - 19, 12r - 17, 12r - 15, 12r - 13, 12r - 11, 12r - 9, 12r - 7, 12r - 5, 12r - 3, 12r - 1, 12r + 1, 12r + 3, 12r + 5, 12r + 7, 12r + 9, \dots, 16r + 3, 16r + 5, 16r + 7, 16r + 9, 16r + 11, \dots, 24r - 27, 24r - 25, 24r - 23, 24r - 21, 24r - 19, 24r - 17, 24r - 15, 24r - 13, 24r - 11, 24r - 9, 24r - 7, 24r - 5, 24r - 3, 24r - 1, 24r + 1, 24r + 3\}$
$= \{1,3,5,7,9, \dots, 24r + 3\}$

obtained different labels on each edge and $f^*\left(E\big(D(r) \cup D(r)\big)\right) = \{1, 3, 5, 7, \dots, 24r + 3\}$ then the edge labeling function $f^*: E\big(D(r) \cup D(r)\big) \rightarrow \{1, 3, 5, 7, \dots, 24r + 3\}$ is bijective. Such that the union of double quadrilateral flower graphs $D(r) \cup D(r)$ with $r \geq 1$ are odd harmonious graphs. ■

The following two examples are given for graph $D(5) \cup D(5)$ in Figure 6 and graph $D(6) \cup D(6)$ in Figure 7, which are odd harmonious graphs.

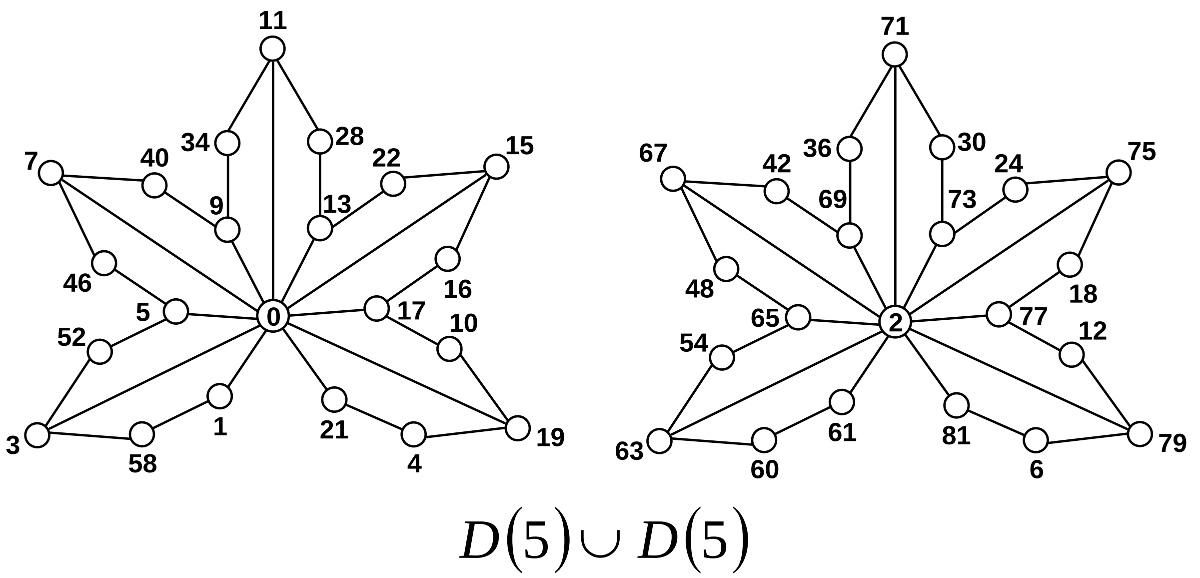

Figure 6. Odd Harmonious Labeling on the Graph $D(5) \cup D(5)$

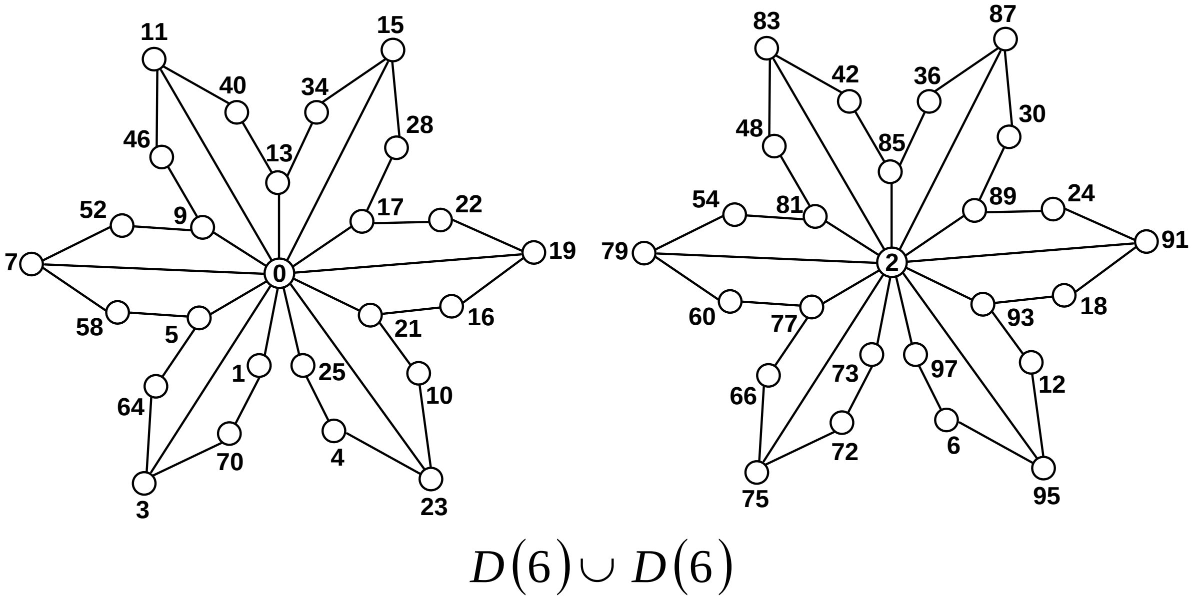

Figure 7. Odd Harmonious Labeling on the Graph $D(6) \cup D(6)$

The definition and construction of various flower graphs are provided in this section. It will also be demonstrated that the rosella flower graphs are odd harmonious graphs.

Definition 3. The rosella flower graphs $R(s)$ with $s \geq 1$ as graph that has vertex set $V(R(s)) = \{a_0\} \cup \{a_j | 1 \leq j \leq 3\} \cup \{b_j | 1 \leq j \leq s\} \cup \{c_j | 1 \leq j \leq s\}$
and edge set

$$E(R(s)) = \{a_0 a_j | 1 \leq j \leq 3\} \cup \{a_1 c_j | 1 \leq j \leq s\} \cup \{a_2 c_j | 1 \leq j \leq s\} \cup \{a_2 b_j | 1 \leq j \leq s\} \cup \{a_3 b_j | 1 \leq j \leq s\}$$

Based on Definition 5, we obtained that order $p = |R(s)| = 2s + 4$ and size $q = |E(R(s))| = 4s + 3$. The following is the construction of the rosella flower graphs in Figure 8.

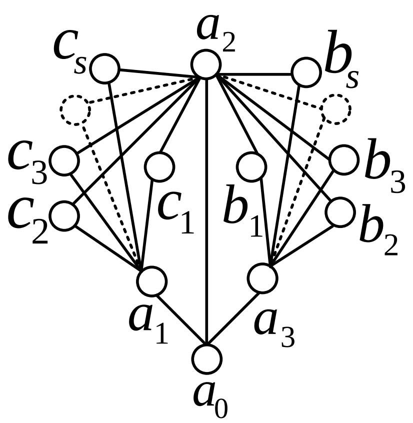

Figure 8. Construction of the Rosella Flower Graphs $R(s)$

Theorem 3. The rosella flower graphs $R(s)$ with $s \geq 1$ are odd harmonious graphs

Proof.
Define the vertex labeling function of graphs $R(s)$ with $s \geq 1$ as follows

$$f(a_0) = 0 \quad (32)$$

$$f(a_j) = 2j - 1, 1 \leq j \leq 3 \quad (33)$$

$$f(b_j) = 8j - 4, 1 \leq j \leq s \quad (34)$$

$$f(c_j) = 8j + 2, 1 \leq j \leq s \quad (35)$$

Based on (32), (33), (34), and (35)

$f\left(V\left(R(s)\right)\right) = \{0\} \cup \{1,3,5\} \cup$
$\{4,12,20,28,36, \dots ,8s-4\} \cup$
$\{10,18,26,34,42, \dots ,8s+2\}$
$= \{0,1,3,4,5,10,12,18,20,26,28,34,36,42, \dots ,8s - 4,8s+2\}$
$= \{0,1,3,4,5, \dots ,8s+2\}$
$\subseteq \{0,1,2,3,4, \dots ,8s+5\}$

obtained different labels on each vertex, and the vertex labeling function $f: V\left(R(s)\right) \to \{0,1,2,3,4, \dots ,8s+5\}$ is injective.
Define the edge labeling function of graphs $R(s)$ with $s \geq 1$ as follows

$$f^*\left(a_0 a_j\right) = 2j-1, 1 \leq j \leq 3 \quad (36)$$
$$f^*\left(a_1 c_j\right) = 8j+3, 1 \leq j \leq s \quad (37)$$
$$f^*\left(a_2 c_j\right) = 8j+5, 1 \leq j \leq s \quad (38)$$
$$f^*\left(a_2 b_j\right) = 8j-1, 1 \leq j \leq s \quad (39)$$
$$f^*\left(a_3 b_j\right) = 8j+1, 1 \leq j \leq s \quad (40)$$

Based on (36), (37), (38), (39), and (40)

$f^*\left(E\left(R(s)\right)\right) = \{1,3,5\} \cup$
$\{11,19,27,35, \dots ,8s+3\} \cup$
$\{13,21,29,37, \dots ,8s+5\} \cup$
$\{7,15,23,31, \dots ,8s-1\} \cup$
$\{9,17,25,33, \dots ,8s+1\}$
$= \left\{\begin{matrix} 1,3,5,7,9,11,13,15,17,19,21, \\ 23,25,27,29,31,33,35,37, \dots, \\ 8s-1,8s+1,8s+3,8s+5 \end{matrix}\right\}$
$= \{1,3,5,7,9, \dots ,8s+5\}$

obtained different labels on each edge and $f^*\left(E\left(R(s)\right)\right) = \{1,3,5,7, \dots ,8s+5\}$ then the edge labeling function $f^*: E\left(R(s)\right) \to \{1,3,5,7, \dots ,8s+5\}$ is bijective. Such that the rosella flower graphs $R(s)$ with $s \geq 1$ are odd harmonious graphs. ■

The following two examples are given for graph $R(5)$ in Figure 9 and graph $R(6)$ in Figure 10 which are odd harmonious graphs.

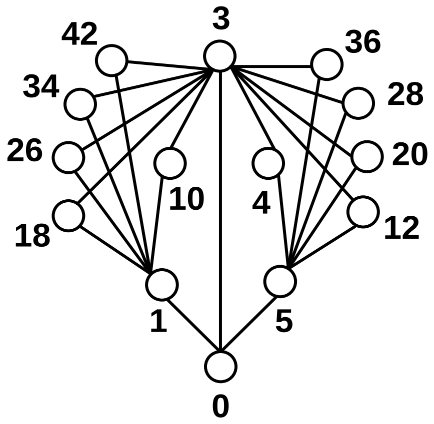

Figure 9. Odd Harmonious Labeling on the Graph $R(5)$

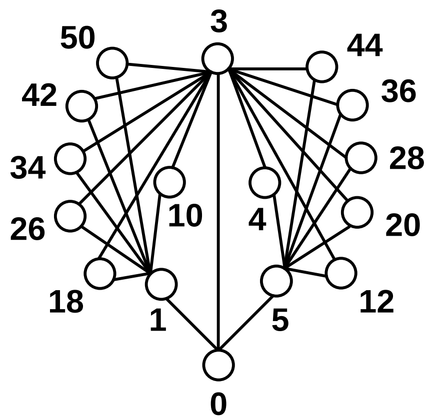

Figure 10. Odd Harmonious Labeling on the Graph $R(6)$

The definition and construction of the union of the rosella flower graphs are provided in this section. It will also be demonstrated that the union of the rosella flower graphs are odd harmonious graphs.

Definition 4. The union of the rosella flower graphs $R(s) \cup R(s)$ with $s \geq 1$ as graph that has vertex set $V\left(R(s) \cup R(s)\right) = \{a_0\} \cup \left\{a_j \middle| 1 \leq j \leq 3\right\} \cup \{b_0\} \cup$
$\left\{b_j \middle| 1 \leq j \leq s-1\right\} \cup \{c_0\} \cup$
$\left\{c_j \middle| 1 \leq j \leq s-1\right\} \cup \{d_0\} \cup$
$\left\{d_j \middle| 1 \leq j \leq 3\right\} \cup \left\{e_j \middle| 1 \leq j \leq s\right\} \cup$
$\left\{g_j \middle| 1 \leq j \leq s\right\}$
and edge set
$E\left(R(s) \cup R(s)\right) = \left\{a_0 a_j \middle| 1 \leq j \leq 3\right\} \cup$
$\{a_1 c_0\} \cup \left\{a_1 c_j \middle| 1 \leq j \leq s-1\right\} \cup \{a_2 c_0\} \cup$

$\{a_2c_j|1 \le j \le s-1\} \cup \{a_2b_0\} \cup$
$\{a_2b_j|1 \le j \le s-1\} \cup \{a_3b_0\} \cup$
$\{a_3b_j|1 \le j \le s-1\} \cup \{d_0d_j|1 \le j \le 3\} \cup$
$\{d_1g_j|1 \le j \le s\} \cup \{d_2g_j|1 \le j \le s\} \cup$
$\{d_2e_j|1 \le j \le s\} \cup \{d_3e_j|1 \le j \le s\}$

Based on Definition 7, we obtained that order $p = |R(s) \cup R(s)| = 4s + 8$ and size $q = |E(R(s) \cup R(s))| = 8s + 6$. The following is the construction of rosella flower graphs in Figure 11.

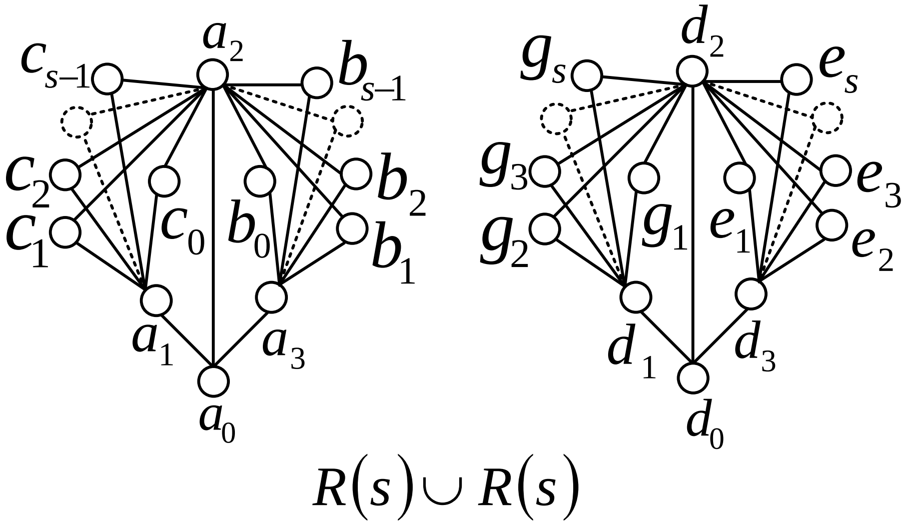

Figure 11. Construction of the Union of the Rosella Flower Graphs $R(s) \cup R(s)$

Theorem 4. The union of the rosella flower graphs $R(s) \cup R(s)$ with $s \ge 1$ are odd harmonious graphs

Proof.

Define the vertex labeling function of graphs $R(s) \cup R(s)$ with $s \ge 1$ as follows

$f(a_0) = 0$ (41)

$f(a_j) = 2j - 1, 1 \le j \le 3$ (42)

$f(b_0) = 4$ (43)

$f(b_j) = 8s + 8j + 10,$
$1 \le j \le s-1$ (44)

$f(c_0) = 10$ (45)

$f(c_j) = 8s + 8j + 16,$
$1 \le j \le s-1$ (46)

$f(d_0) = 2$ (47)

$f(d_j) = 2j + 11, 1 \le j \le 3$ (48)

$f(e_j) = 8j - 2, 1 \le j \le s$ (49)

$f(g_j) = 8j + 4, 1 \le j \le s$ (50)

Based on (41), (42), (43), (44), (45), (46), (47), (48), (49), and (50)

$f\left(V(R(s) \cup R(s))\right) = \{0\} \cup \{1,3,5\} \cup \{4\} \cup \{8s + 18, 8s + 26, 8s + 34, 8s + 42, \dots, 16s + 2\} \cup \{10\} \cup \{8s + 24, 8s + 32, 8s + 40, 8s + 48, \dots, 16s + 8\} \cup \{2\} \cup \{13,15,17\} \cup \{6,14,22,30, \dots, 8s - 2\} \cup \{12,20,28,36, \dots, 8s + 4\}$
$= \{0,1,3,4,5, 8s + 18, 8s + 24, 8s + 26, 8s + 32, 8s + 34, 8s + 40, 8s + 42, 8s + 48, \dots, 16s + 2, 16s + 8\} \cup \{2,6,13,14,15,17,20,22,28,30,36, \dots, 8s - 2, 8s + 4\}$
$= \{0,1,2,3,4,5,6,13,14,15,17,20, \dots, 8s - 2, 8s + 4, 8s + 18, 8s + 24, 8s + 26, 8s + 32, 8s + 34, 8s + 40, 8s + 42, 8s + 48, \dots, 16s + 2, 16s + 8\}$
$= \{0,1,2,3,4,5, \dots, 16s + 8\}$
$\subseteq \{0,1,2,3,4, \dots, 16s + 11\}$

obtained different labels on each vertex, and the vertex labeling function $f: V(R(s) \cup R(s)) \to \{0,1,2,3,4, \dots, 16s + 11\}$ is injective.

Define the edge labeling function of graphs $R(s) \cup R(s)$ with $s \ge 1$ as follows

$f^*(a_0a_j) = 2j - 1, 1 \le j \le 3$ (51)
$f^*(a_1c_0) = 11$ (52)
$f^*(a_1c_j) = 8s + 8j + 17, 1 \le j \le s - 1$ (53)
$f^*(a_2c_0) = 13$ (54)
$f^*(a_2c_j) = 8s + 8j + 19, 1 \le j \le s - 1$ (55)
$f^*(a_2b_0) = 7$ (56)
$f^*(a_2b_j) = 8s + 8j + 13, 1 \le j \le s - 1$ (57)
$f^*(a_3b_0) = 9$ (58)
$f^*(a_3b_j) = 8s + 8j + 15, 1 \le j \le s - 1$ (59)
$f^*(d_0d_j) = 2j + 13, 1 \le j \le 3$ (60)
$f^*(d_1g_j) = 8j + 17, 1 \le j \le s$ (61)
$f^*(d_2g_j) = 8j + 19, 1 \le j \le s$ (62)
$f^*(d_2e_j) = 8j + 13, 1 \le j \le s$ (63)
$f^*(d_3e_j) = 8j + 15, 1 \le j \le s$ (64)

Based on (51), (52), (53), (54), (55), (56), (57),(58), (59), (60), (61), (62), (63), and (64)

$f^*\left(E\left(R(s) \cup R(s)\right)\right) = \{1,3,5\} \cup \{11\} \cup \{8s + 25, 8s + 33, 8s + 41, 8s + 49, \dots, 16s + 9\} \cup \{13\} \cup \{8s + 27, 8s + 35, 8s + 43, 8s + 51, \dots, 16s + 11\} \cup \{7\} \cup \{8s + 21, 8s + 29, 8s + 37, 8s + 45, \dots, 16s + 5\} \cup \{9\} \cup \{8s + 23, 8s + 31, 8s + 39, 8s + 47, \dots, 16s + 7\} \cup \{15,17,19\} \cup \{25,33,41,49, \dots, 8s + 17\} \cup \{27,35,43,51, \dots 8s + 19\} \cup \{21,29,37,45, \dots, 8s + 13\} \cup \{23,31,39,47, \dots, 8s + 15\}$
$= \{1,3,5,7,9,11,13, 8s + 21, 8s + 23, 8s + 25, 8s + 27, 8s + 29, 8s + 31, 8s + 33, 8s + 35, 8s + 37, 8s + 39, 8s + 41, 8s + 43, 8s + 45, 8s + 47, 8s + 49, \dots, 16s + 5, 16s + 7, 16s + 9, 16s + 11\} \cup \{15,17,19,21,23,25,27,29,31,33,35, \dots, 8s + 138s + 15, 8s + 17, 8s + 19\}$
$= \{1,3,5,7,9,11,13,15,17,19, \dots, 8s + 138s + 15, 8s + 17, 8s + 19, 8s + 21, 8s + 23, 8s + 25, 8s + 27, 8s + 29, 8s + 31, 8s + 33, 8s + 35, 8s + 37, 8s + 39, 8s + 41, 8s + 43, 8s + 45, 8s + 47, 8s + 49, \dots, 16s + 5, 16s + 7, 16s + 9, 16s + 11\}$
$= \{1,3,5,7,9, \dots, 16s + 11\}$

obtained different labels on each edge and $f^*\left(E\left(R(s) \cup R(s)\right)\right) = \{1, 3, 5, 7, \dots, 16s + 11\}$ then the edge labeling function $f^*: E\left(R(s) \cup R(s)\right) \to \{1, 3, 5, 7, \dots, 16s + 11\}$ is bijective. Such that the rosella flower graphs $R(s) \cup R(s)$ with $s \ge 1$ are odd harmonious graphs. ■

The following two examples are given for graph $R(5) \cup R(5)$ in Figure 12 and graph $R(6) \cup R(6)$ in Figure 13, which are odd harmonious graphs.

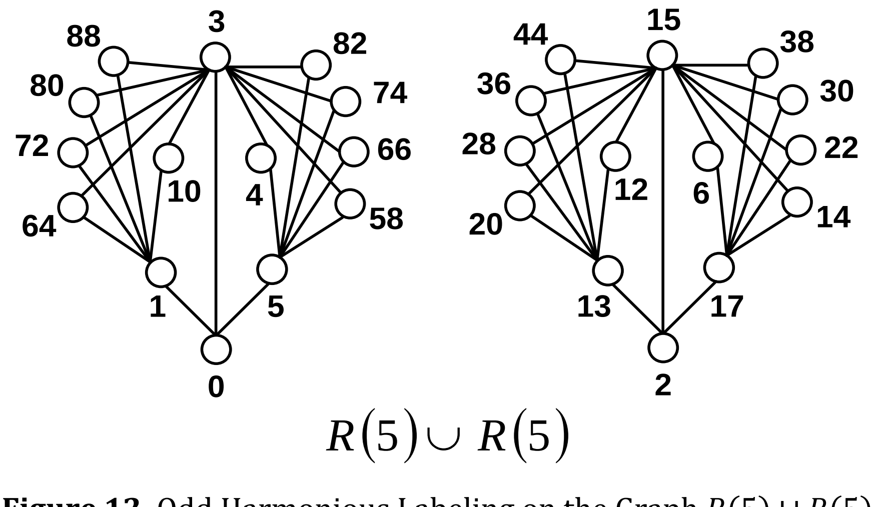

Figure 12. Odd Harmonious Labeling on the Graph $R(5) \cup R(5)$

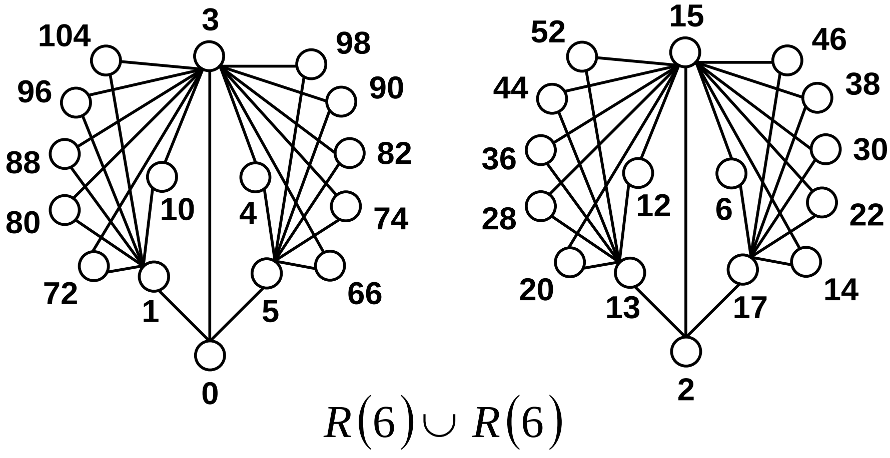

Figure 13. Odd Harmonious Labeling on the Graph $R(6) \cup R(6)$

Based on the results in Definition 1, it is obtained that the zinnia flower graph class found by Firmansah et al. (2023) has been successfully developed into a new graph class in the form of a union of zinnia flower graphs. Furthermore, based on the result in Theorem 1, it is proven that the union of zinnia flower graphs is an odd harmonious graph.

On the other hand, based on the results in Definition 2, it is obtained that the double quadrilateral flower graph class found by Firmansah (2020) has been successfully developed into a new graph class in the form of a combined double quadrilateral flower graph. Furthermore, based on the result in Theorem 2, it is proven that the union of double quadrilateral flower graphs is an odd harmonious graph.

Based on the discussion, the construction and definition of the rosella flower graph in Definition 3 and the union of the rosella flower graph in Definition 4 are obtained. On the other hand, it has also been proven that the rosella flower graph and the union of the rosella flower graph are odd harmonious graphs stated in Theorem 3 and Theorem 4.

Based on these results, a novelty has been obtained in the form of a new graph class, namely the union graph of the zinnia flower graph, the union graph of the double quadrilateral flower graph, the rosella flower graph, and the union graph of the rosella flower graph. Furthermore, it has been proved that the new graph class satisfies the odd harmonious labeling property so that it belongs to the odd harmonious graph family.

CONCLUSIONS AND SUGGESTIONS

Based on the theorems proved, it is found that the class of graphs under study satisfies an injective vertex labeling function that induces a bijective edge labeling such that it satisfies the properties of odd harmonious labeling. Furthermore, it is found that the union of zinnia flower graphs, the union of double quadrilateral flower graphs, the rosella flower graphs, and the union of the rosella flower graphs are odd harmonious graphs.

This research can be continued by developing zinnia flower graphs, double quadrilateral flower graphs, and the rosella flower graphs with other graph operations.

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