Desimal: Jurnal Matematika Vol 7 No 3 (2024) 463-474

Contents lists available at DJM

DESIMAL: JURNAL MATEMATIKA

p-ISSN: 2613-9073 (print), e-ISSN: 2613-9081 (online), DOI 10.24042/djm

http://ejournal.radenintan.ac.id/index.php/desimal/index

High school students' justification to solve algebraic mathematics reasoning problems: Descriptive analysis from the nature of justification tasks

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ARTICLE INFO

Article History

Received : 19-07-2024
Revised : 09-08-2024
Accepted : 22-08-2024
Published : 06-11-2024

Keywords:
Algebraic Problems; Justification; Mathematics Reasoning; Problem-Solving; Types of Justification.

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Doi: 10.24042/djm.v7i3.23504

ABSTRACT

Justification serves as a tool used to improve students' ability to understand mathematics and their proficiency in working on mathematical problems. However, despite its significance, student justification in the problem-solving process has not become a priority for teachers based on several studies. While justification research related to problem-solving has begun to develop, it is only limited to validating the truth of a mathematical solution. Thus, this qualitative descriptive research aims to analyze students' justification process in solving reasoning problems regarding the types of justification (interpretation, elaboration, prediction, and validation) and the function of each type. The research subjects consisted of two high school students in Indonesia with high abilities who solved algebraic mathematics reasoning problems. Meanwhile, data collection and analysis used the results of students solving algebraic reasoning problems involving the nature of justification tasks. The results show that the types of justification indicate a crucial role in students' problem-solving process. Apart from that, each of these also has the potential for use in the problem-solving process. Furthermore, this article also suggests several points that can be applied to develop student justification in reasoning algebraic problems.

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INTRODUCTION

In the context of learning practice, justification is a tool used to improve students' ability to understand mathematics and their proficiency in working on mathematical problems (Staples, Bartlo, & Thanheiser, 2012; Supriani, Fardillah, Rmudi, & Herman, 2019). In justifying, students are asked to provide mathematical reasons and express a more complex and higher level of mathematical cognitive process, as noted by Hiebert et al. (1997) and Wood, Williams, & McNeal (2006). This process encourages students to explain their problem-solving strategies and solutions, as well as facilitate understanding of mathematics convincingly. Through student involvement in justification, they are invited to explain their solutions mathematically, thereby encouraging a deeper understanding of mathematical concepts, articulating thought processes, and validating conclusions through systematic reasoning. In addition, justification instills mathematical fluency processes and empowers students to communicate problem solutions effectively and persuasively (Mata-Pereira & da Ponte, 2017).

Simon & Blume (1996) stated that the process of proving something valid and developing an argument based on the available evidence is justification. According to Pritchard & Neta (2008), a justification is a set of answers or explanations provided by an individual in response to a question about why they arrived at a particular mathematical conclusion. In addition, eliminating one's doubts is a necessary step in discovering the truth, but persuasion entails doing the same for others (Ellis, 2007).

Justification is closely related to reasoning, which is one of the 21st-century necessary competencies to face current developments. A student will encounter various problems both at school and outside school, so the ability to justify the situation is an important element. Generally, reasoning is a process to conclude by taking all related factors into account (Umay, 2003). In addition, reasoning is the ability to analyze mathematical situations and construct logical arguments (Chua, 2016). Justification, in a broad sense, is defined as the actions students take to explain to others and themselves when they encounter a situation, what they see, what they do, what they think, and why they do it (Hershkowitz, 2020).

Justification of the solution is crucial as a student's goal in working on mathematical problems (Prabawanto, 2019). Several recent studies show great attention to students' thinking, which leads to creative differences in problem-solving approaches and students' explanations of solutions to problems proposed (Hidajat, 2021; Islam, Budiyono, & Siswanto, 2021; Khalid et al., 2020). This increased focus comes from the realization that many students believe that all mathematical problems can be solved in a short time and they will not survive if the problem is solved for a long time (Glass & Maher, 2004; Phonapichat, Wongwanich, & Sujiva, 2014).

The prominent problem-solving stages by Polya (1957) stated that there are four stages of problem-solving including understanding the problem, making a plan, carrying out the plan, and looking back. In the initial phase, students must clearly perceive the requirements of the problem and discern the connections between various elements within it. Subsequently, in the second and third stages, they must ascertain how the unknown factors in the problem relate to the known data, enabling them to formulate a solution strategy and execute it accordingly. Finally, in the concluding stage, individuals need to retrospectively evaluate the completed solution to determine its accuracy.

Currently, there is a general dearth of justification in mathematics classes (Jacobs et al., 2006), even when teachers are carrying out evidence-related tasks (Bieda, 2010; Store, 2015). This condition is truly worrying because teachers have not been able to present problem-solving tasks that explore students' justifications. The justification task is the key to providing valid information about students' mathematical abilities. Apart from that, the presence of justification tasks in the mathematics school community is important to fulfill the vision of mathematics learning as a tool to support the thinking process.

The role of justification in the literature is elaboration and validation following the definition of justification that has been presented. Thus, justification tasks are explained as questions that ask students to elaborate, explain, and validate mathematical results. Furthermore, (Chua, 2016) developed four justification tasks into four categories, namely elaboration, interpretation, prediction, and validation. Each nature of justification task has different demands. This framework will be used in this study presented in Table 1, which provides an overview of the four types of justification tasks and aims in each type.

Table 1. Nature of Justification Tasks

Nature of Justification Tasks	Examples	Aims
Elaboration	Explain how… (and so on).	Clearly explain the method or strategy employed to achieve the mathematical result.
Interpretation	Explain what… (and so on).	Provide the interpretation or significance of the mathematical result.
Prediction	Explain whether… (and so on). Explain which… (and so on).	Decide the mathematical claim and provide evidence to support or refute the claim.
Validation	Explain which… (and so on).	Give reason or evidence to support the mathematical claim.

Source: (Chua, 2016)

Furthermore, research on justification in solving problems has begun to be developed by several studies. (Glass & Maher, 2004) researched the form of student justification in solving combinatorics problems. Apart from that, (Stylianou, 2013) also analyzed the relationship between the justification process and mathematical representation in problem-solving. Most studies on this topic analyze justification as a validation of students' answers, even though the current justification relationship is not limited to just validating answers but has developed into other forms such as interpretation, elaboration, prediction, and validation (Chua, 2016). In addition, the problem is also not specific to a particular thinking process, even though justification has a close relationship with reasoning. Thus, this study will discuss the process of each type of justification in solving students' reasoning problems. Moreover, each type of justification will be analyzed about the process of solving mathematical problems. In addition, it is essential to provide suggestions for developing students' justifications in the reasoning problem-solving process.

METHOD

Descriptive qualitative research is the method used in this research. The purpose of this research was to explore the subject's justification for solving algebraic reasoning problems. Purposive sampling was the technique to select two subjects as samples for this study. Subjects are the high mathematics ability levels in a class. They were students from a 10th grade senior high school in Indonesia. Mathematics justification can be stimulated by tasks specifically designed to know how students justify the process to get the final solution. The procedure of research includes: 1) the researcher selects two students who have high mathematical abilities as sampling based on daily examination in the class (to make sure the students can solve the problem);

2) the researcher gives a task about mathematics algebraic problem that has been prepared to provide the student's justification (time: 1 hour); 3) the tasks have two problems that contain four types of justification, including interpretation, elaboration, prediction, and validation; and 4) the researcher analyzes the answer based on each type of justification. The data collection flow is illustrated in Figure 1.

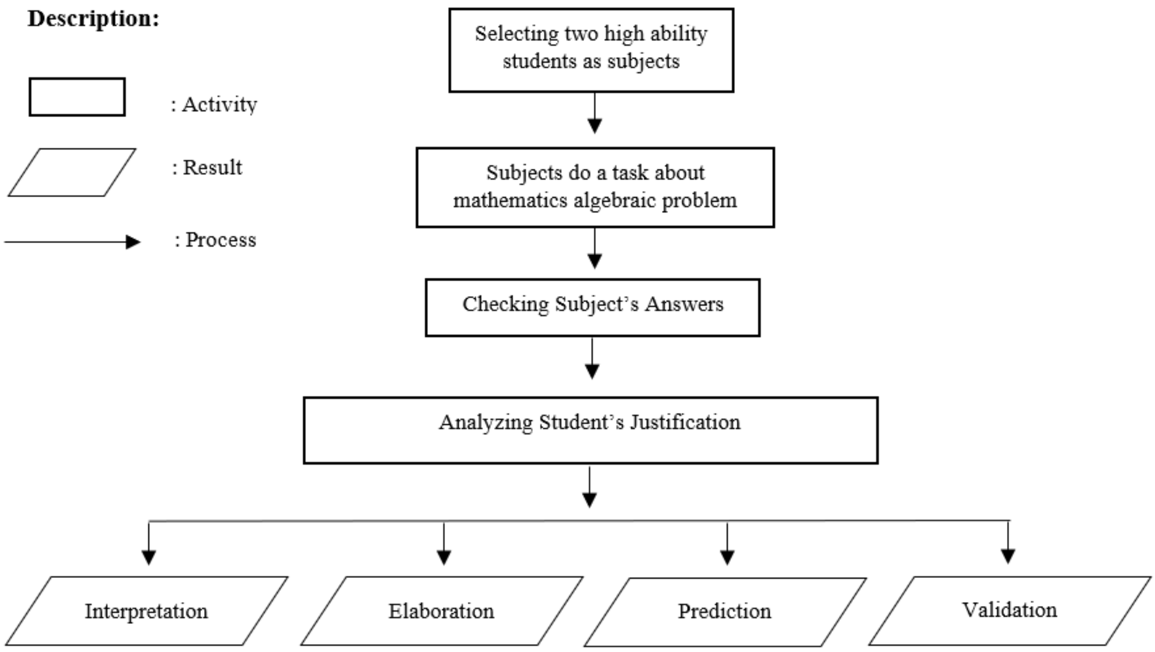

Figure 1. Flowchart of Collecting Data

The task below includes two algebraic problems whose topics students have studied at previous levels, as follows:

(Please answer in detail and provide reasons for each of your answers)
Time: 1 hour

Problem 1: Two-Variable Linear Equation Systems
Boni has an interesting question and tries to answer it. He was asked to determine the age of a father and his son in 2020 from a story problem. To solve it, he took the following steps:
a. Boni supposes that if the father's age = x and the son's age = y. Next, he correctly modeled the relationship between the father's age and the son's age in 2012 to form $x - 8 = 6(y - 8)$*. What does this equation mean?*
b. Boni also correctly modeled the relationship between the father's age and the child's age in 2016 to form $x - 4 = (y - 4) + 25$*. What does this equation mean?*
c. With the two equations that Boni obtained, help him determine the ages of the father and the child in 2020.

Problem 2: Arithmetic Sequences
There are two meeting rooms, namely room A and room B, on the Dimas campus. Each meeting room consists of twenty rows of chairs. In room A, there are 10 seats in the first row, which always increases by 3 for each row. Meanwhile, in room B, there are 20 seats.
In the first row, and it always increases by 2 for each row.
Dimas stated that:

a. There are 34 seats in the 9th row in room A and 32 seats in the 7th row in room B. Is Dimas' statement correct?
b. In the same row in rooms A and B, there is the same number of seats. If this statement is true, what sequence is Dimas referring to?

RESULTS AND DISCUSSION

In Problem 1, students are tasked to interpret a mathematical result and elaborate on the process to achieve the final mathematical result. Initially, students tried to interpret the mathematical model presented in question 1a and 1b. Students will understand the problem by looking at the mathematical model in which ‘x’ is the variable of the father’s age and ‘y’ is the variable of the son’s age in 2020. In the year 2012, a relation between the father’s and son’s age represented by a mathematical model is $x - 8 = 6(y - 8)$, indicating eight years ago, the father’s age was six times the son’s age. Then, the mathematical model according to the relation in the year 2016 is $\mathrm{x} - 4 = (\mathrm{y} - 4) + 25$, which means four years ago, the father’s age is twenty-five years older than the son’s age. The works of Student A and Student B can be seen in Figures 2 and 3.

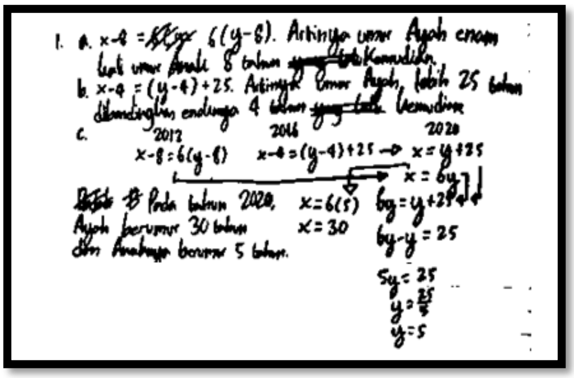

Figure 2. Student A’s Work on Problem 1

Student A initially supposed that it meant eight years later, the father’s age is six times the son’s age (as in the answer 1a), and the father’s age is more than twenty-five years older than the son’s age (as in the answer 1b). However, student A later became uncertain about the statement and crossed out the word “ago” in the answer. Student A's answer did not consider the relationship between last year's age and the current year because student A wrongly interpreted the mathematics model. Otherwise, student B still wrote the mathematics model to “if x is subtracted by 8 (because of 2020-2012), equal 6 times y and subtracted by 8 (as in the answer 1a), and if father’s age is subtracted by 4 (because of 2020-2016), equal son’s ages subtracted by 4 and added by 25” (as in the answer 1b). Then, student B made a mistake in comprehending the question that asked the interpretation, leading to an inadequate justification. These instances illustrate ongoing difficulties faced by students to recognize mathematics problems clearly and interpret mathematical models to justify their reasoning.

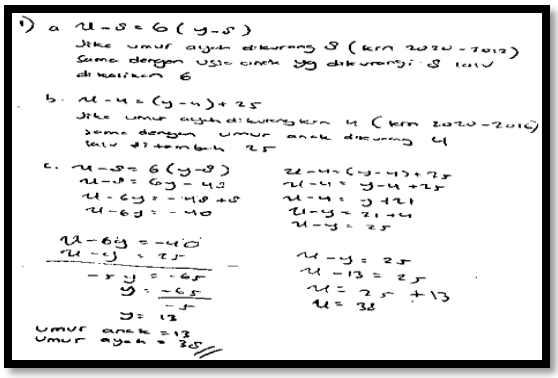

Figure 3. Student B’s Work on Problem 1

Interpretation mathematics tasks present significant challenges for the students. If students have problems understanding the question, it will cause some errors in solving the problem. As did student B, the result indicated the student was not used to justifying the form of interpretation. Understanding the problem is the first and most important step to solving the problem (Polya, 1957). Thus, having clear justification will help students make a comprehensive understanding and explain their thinking processes. As noted by Amen (2006), the comprehension of tasks, especially in solving word problems, is determined by an understanding of mathematical vocabulary, and the lack of it will influence capabilities to solve problems.

The students employed different strategies to elaborate on the solution (as in answer 1c). Student A used the substitution, and student B used the mix method (substitution and elimination). Even though the operations carried out were correct, student A made a mistake in interpreting the mathematical model, resulting in changes to the mathematical model that led to an incorrect solution. Student A changed the mathematical model to $x = 6y$ and $x = y + 25$ in 2020, which is a false claim. Then, student A did substitution $x = 6y$ to $x = y + 25$. That's why student A got the solution $x = 30$ and $y = 5$ and concluded that the father's age is 30 and the son's age is 5. Student B simplified the model with the operation and got the model $x - 6y = 40$ and $x - y = 25$. The elimination method was used to get the solutions x and y. With the correct mathematical operation, student B found the solution $x = 38$ and $y = 13$ and stated that the father's age is 38 and the son's age is 13. This comparison highlights that understanding the problem differently can produce distinct problem solutions, with student B's elaboration leading to the correct solution.

Mathematical operations and procedures are implemented very well through the students' answers. The students who are stressed intend to use this formula, even though this formula is not the student's single choice to solve the problem. However, mathematics is viewed as a set of interconnected concepts and not isolated procedures. Through learning mathematics, students can explore, justify, and communicate mathematics effectively. This process encourages students to develop depth of concepts and increases reasoning abilities among students. Mathematics learning outcomes develop conceptual depth, procedural flexibility, and reasoning among students. Interaction between teachers and students in mathematics classes is important to achieve these goals (Brodie, 2010).

In Problem 2, students were asked to predict the pattern and validate the statement in the problem. In problem 2a, students are expected to be able to predict statements about number patterns. Both student A and student B understand that the problem is about an arithmetic sequence. Consequently, they used the formula $U_n = a + (n - 1)$ to predict the pattern. Then, they determined that $a_A = 10$, $b_A = 3$, and $a_B = 10$, $b_B = 3$, which substituted to the formula. They found that the ninth sequence in room A has 34 chairs, and room B has 32 chairs in the seventh sequence. The works of student A and student B are explained in Figure 4 and Figure 5. This shows their compliance with formal mathematical procedures in solving mathematical problems.

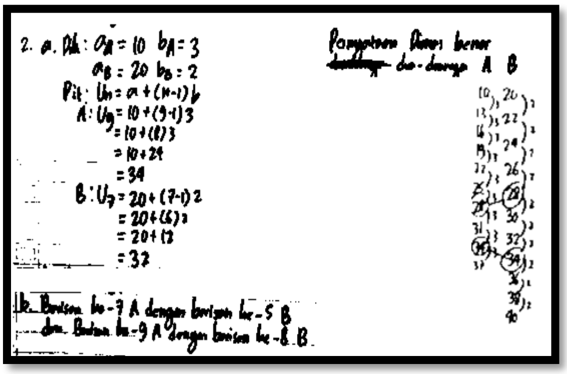

Figure 4. Student A's Work on Problem 2

On the other hand, students are accustomed to solving calculation problems without explanation, only correct and quick answers, a context that may encourage the use of more formal representations or formulas. This shows a general tendency among students to rely on approximate formulas when faced with sequence problems requiring predicting mathematics claims. Such tendencies underscore the existing reliance on procedural approaches rather than conceptual understanding in predicting claims in mathematical problems.

In problem 2b, students are asked to validate a claim regarding the same pattern between two sequences. Student A made a little mistake in understanding the question, which caused it to be wrong to validate a statement. This happened because student A thought that the problem asked for just the same number of chairs in room A and room B. Then, student A counted one by one each sequence until getting the result of the seventh sequence in room A, the fifth sequence in room B, followed by the ninth sequence in room A, and the eighth sequence in room B. Meanwhile, student B found its value in the general form of the sequence $U_n = 3n + 7$ in the sequences of room A and $U_n = 2n + 18$ in the sequences of room B. Then, student B realized that in the same sequence, the number of seats would also be the same, represented by $3n + 7 = 2n + 18$. Therefore, the sequence that has the same number as the chair is the eleventh.

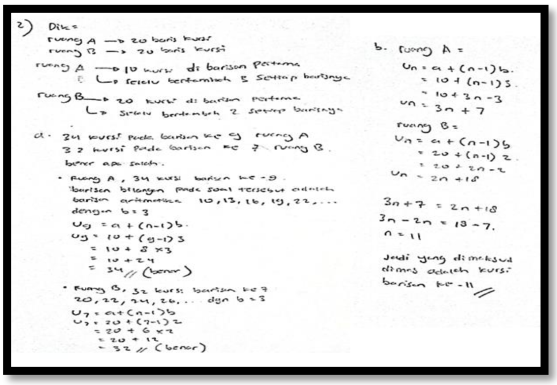

Figure 5. Student B's Work on Problem 2

Students have to provide sufficient evidence to validate a statement (Chua, 2016). This means students not only arrive at a solution but also demonstrate clear reasoning and evidence to demonstrate the validity of their explanation; just getting numerical results is not enough to validate the solution. Students need to articulate the logic or reasoning behind their solutions. This is in line with the principle of mathematical reasoning, which evolves in the process of finding solutions and justifying them logically.

In summary, the students have made justifications by their respective thought processes. Each task faced by students will ask for various types of justification. Instructional tasks that are designed to raise questions about the viability of various generalization strategies and the validity and power of various justifications (Lannin, 2005). When students can provide correct justification, they can not only solve problems well but also improve their process of thinking and mathematical communication.

Interpretation is an important aspect of solving mathematical problems because it involves deciphering the meaning behind mathematical results in the context of a particular problem (Stillman, Blum, & Biembengut, 2015). In the context of reasoning problems, just like other thinking processes, students are not only asked to understand the mathematical model presented but must be able to interpret the model or vice versa. In the solving problem process, both students A and B tried to interpret the mathematical model of the problem. Both students seemed to struggle to understand the problem well based on their interpretation. In addition, student A had difficulty understanding the problem, so she misinterpreted the mathematical model interpretation. On the other hand, student B also restated the mathematical model without interpreting it in the context of the problem. This is in line with (Phonapichat et al., 2014), who stated that students have difficulty understanding keywords in problems and have difficulty determining what is needed to solve the problem. Thus, the process of providing valid solutions in problem-solving requires students' ability to interpret mathematical information (Intaros, Inprasitha, & Srisawadi, 2014; Prabawanto, 2019).

Elaboration in problem-solving shows students' ability to use methods or strategies to obtain solutions (Bayazit, 2013). In addition, students' ability to carry out elaboration is closely related to procedural fluency in solving problems (Andal & Andrade, 2022; Inayah, Septian, & Suwarman, 2020). Then, the fluency of procedures consists of strategic implementation, something that is emphasized in solving problems. Although this process initially requires a proper understanding of the problem, both student A and student B showed good algebraic manipulation skills in solving reasoning problems. Apart from that, the two students also succeeded in elaborating using various methods, although the approach used by student B was more efficient and accurate because student A was wrong in interpreting the mathematical model. However, in an effort to carry out elaboration, many students use memorized formulas as a shortcut to carry out problem-solving strategies, but this becomes a habit that students often do and sometimes hinders students' mathematical thinking processes (Hewitt, 2011). Therefore, in elaboration, even though students used to apply certain mathematics formulas, maintaining a balance between procedural fluency and understanding with thinking ability becomes very important in problem-solving skills.

Predictions in problem-solving are related to making decisions regarding a mathematical claim and providing evidence for a claim (Lim, Buendía, Kim, Cordero, & Kasmer, 2010). When making predictions, students tend to predict a pattern logically and systematically. In solving problems, student A and student B both demonstrated proficiency by directly using formulas to predict patterns in arithmetic sequences. Both students demonstrated a good understanding of arithmetic sequences and used appropriate formulas. Even though students are used to applying mathematical formulas, the prediction task highlights students' understanding of choosing the right strategy through understanding the problem. As long as students succeed in predicting patterns from mathematical sequences and provide clear explanations for their predictions, this will improve students' problem-solving abilities and deepen their understanding of the concepts involved. Success in predicting patterns from mathematical sequences, coupled with the ability to provide a coherent explanation of these predictions, will make a significant contribution to students' problem-solving skills and deepen students' understanding of the concepts involved (Kasmer & Kim, 2011).

Validation is related to providing reasons for validating statements or confirming the accuracy of a mathematical solution (Prabawanto, 2019). In the problems, students are asked to confirm existing solutions. Students validate whether their answers are mathematically correct and make sense. Both students tried to validate the statement, but student A paid less attention to systematic methods. Wrong understanding leads to wrong validation. Lack of adherence to systematic procedures can lead to validation errors because it ignores potential inaccuracies or inconsistencies in the solution process. Student B's approach is more effective and aligned with the needs of the problem, thereby demonstrating a strong understanding of problem-solving techniques. This depiction underscores the important role of methodological rigor in validation tasks, as it not only ensures the accuracy of solutions but also fosters a deeper understanding of mathematical concepts and problem-solving strategies (Ko & Knuth, 2013).

Furthermore, each type of justification has a close relationship with the problem-solving process. Recently, justification cannot be categorized as part of the problem-solving process, but with the development of justification types, this practice has become an important part of problem-solving. The interpretation process is related to how students interpret a mathematical model or result. Based on this study, it has the potential to strengthen students' understanding of mathematical problems, which is also in accordance with a research study conducted by Ramdhani, Usodo, & Subanti (2017). Then, the process of prediction is related to the ability to make decisions or claims, which probably emphasizes strategic planning in problem-solving (Nuryadi & Hartono, 2021). This is in accordance with (Matteson, Capraro, Capraro, & Lincoln, 2012), who stated that the prediction process is a key part of problem-solving.

Then, elaboration is the ability to explain methods and strategies for solving problems. This might help in the process of implementing problem-solving strategies. This is in line with research conducted by (Intaros et al., 2014), who stated that in solving problems, students use different strategies and approaches to find solutions. Then, the validation process is related to checking whether a solution is correct (Prabawanto, 2019). This process is related to checking the understanding of the problem and the strategies that have been implemented to ensure the correctness of the solution to the problem. The types of justification based on the problem-solving process of students can be seen in Figure 6.

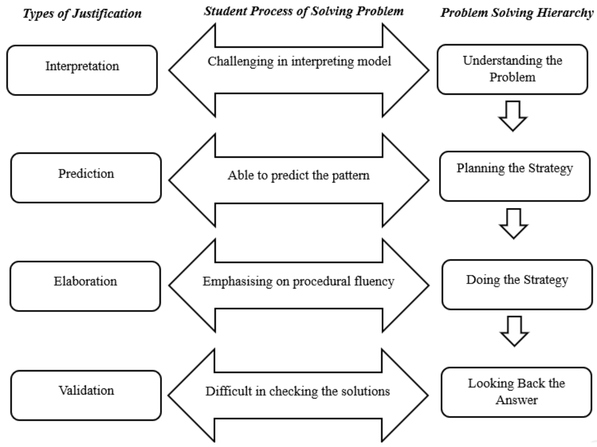

Figure 6. Types of Justification based on the Student Problem-Solving Process

CONCLUSIONS AND SUGGESTIONS

Each type of student justification plays an important role in the problem-solving process. The data shows that each student's justification ability from the type has a different function in the problem-solving process. During the interpretation process, students show efforts in interpreting the model presented in reasoning problems, but they may face challenges. This phase is closely related to the process of understanding students' problems in solving problems. Subsequently, when elaborating, students demonstrate the ability to derive strategies to obtain solutions, emphasizing procedural fluency. Furthermore, in the prediction phase, students are able to plan strategies appropriately related to number patterns. Lastly, in validating the solution, one student also had difficulty checking the solution, and another student showed a strong understanding of checking the information needed and problem-solving strategies to get a solution.

This study can be explored further for samples with a wider range of abilities because its limitation is that it only uses a small sample with homogeneous abilities. Thus, the justifications used by students in solving problems are also more varied. In addition, the strategy of incorporating justification in mathematics learning also still needs to be investigated.

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