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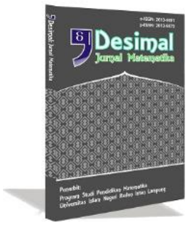

Intuitionistic regular subspaces in intuitionistic topological spaces

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ABSTRACT

This paper explores the concept of intuitionistic topological spaces, delving into their definitions and essential properties. It also examines intuitionistic topological subspaces, providing insights into their characteristics. Additionally, the paper investigates intuitionistic regular spaces and demonstrates their hereditary nature, specifically focusing on $R(i), R(ii), R(iii)$, *and* $R(iv)$. *To illustrate these concepts in practical terms, the paper presents two real-world examples of intuitionistic sets. Through a comprehensive analysis of intuitionistic topological spaces and their subsets, the study sheds light on the inheritability of regularity in these spaces. Furthermore, the work emphasizes the significance of intuitionistic topological spaces within the realm of mathematical research, showcasing their applicability through concrete instances of intuitionistic sets.*

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INTRODUCTION

The development of the field of intuitionistic topological spaces has unfolded through a sequence of seminal contributions by various researchers. It commenced in Zadeh (1965), with Zadeh's pioneering work on fuzzy sets, establishing the initial groundwork for this domain. In 1983, Atanassov introduced the concept of intuitionistic fuzzy sets, adding a layer of complexity to the theory. Subsequently, Çoker (1996) introduced intuitionistic sets and intuitionistic points, opening up new avenues of exploration. Building on this foundation, Çoker's (1997) work defined intuitionistic fuzzy topological spaces, and in Çoker (2000), he extended this framework to intuitionistic topological spaces. The year 2001 witnessed the formulation of the T_1 and T_2 separation axioms in intuitionistic topological spaces by Bayhan & Çoker (2001), further enriching the theoretical landscape. In 2012, Yaseen & Mohammad introduced the concepts of Regular and Weak Regular intuitionistic topological spaces, followed by Jassim's (2013) work, which defined completely normal and weakly normal intuitionistic topological spaces. The progression continued with Selvanayaki & Ilango (2015), who introduced IGPR continuity and compactness in intuitionistic topological spaces, and Selvanayaki & Ilango (2016) explored homeomorphism within these spaces. Additionally, Ilango & Albinaa (2016) and Kim, Lim, Lee, & Hur (2017) delved into the properties of intuitionistic closures and interiors in intuitionistic topological spaces, respectively. In the research of Prova & Hossain (2020), they introduced intuitionistic fuzzy-based regular and normal spaces, while (Chae, Kim, Lee, & Hur, 2020) harnessed interval-valued intuitionistic sets for topological applications. Islam, Hossain, & Mahbub (2021) contributed by defining notations on intuitionistic fuzzy r-regular spaces. Most recently, Haque, Akhter, & Murshed (2022) introduced the concept of intuitionistic subspace topology along with an exploration of its properties. Significantly, none of the previous works addressed intuitionistic regular subspaces and their properties. This paper bridges this gap by leveraging the framework of intuitionistic subspace topology, as defined by Haque et al. (2022), and unveiling a groundbreaking insight: the hereditary nature of intuitionistic regular spaces $R(i), R(ii), R(iii)$, and $R(iv)$. This revelation extends the frontiers of our understanding of intuitionistic topological spaces, introducing a new dimension to the discourse within this burgeoning field.

METHOD

The definitions of intuitionistic set and intuitionistic point are given (Çoker, 1996) as follows:

"**Definition 1** (Çoker, 1996)**:** Consider two subsets A_T and A_F of a non-empty set X such that $A_T \cap A_F = \emptyset$. Let $A = (A_T,\ A_F)$, then A is called an intuitionistic set (in short, IS) of X. Here A_T is called the set of members, and A_F is called the set of nonmembers of A."

In fact, A_T is a subset of X agreeing or approving of a certain opinion, view, suggestion, or policy, and A_F is a subset of X refusing or opposing the same opinion, view, suggestion, or policy, respectively.

Consider $\emptyset_I = (\emptyset,\ X)$ and $X_I = (X, \emptyset)$, where $\emptyset_I$ is the intuitionistic empty set and X_I is the intuitionistic whole set of X. In general, $A_T \cup A_F \neq X$. The set of all ISs of X is denoted by IS(X).

"**Definition 2** (Çoker, 1996)**:** Consider an IS A of a non-empty set X and $a \in X$.
(i) $a_I = (\{a\}, \{a\}^c)$ is an intuitionistic point (IP) and $a_{IV} = (\emptyset, \{a\}^c)$ is a vanishing point of X.
(ii) If $a \in A_T$, then $a_I \in A$ and if $a \notin A_F$ then $a_{IV} \in A$."

IP *(X)* is the set of all intuitionistic points or intuitionistic vanishing points in X.

The definition of intuitionistic topological subspace is given in Haque et al. (2022) as follows:

"**Definition 3** (Haque et al., 2022): Let (X, τ) be an ITS and A be a subset of X. The class $\tau_A = \{(G_T \cap A,\ G_F \cap A) \colon G = (G_T, G_F) \in \tau\}$ is an IT on A. Then the IT τ_A is called the relative IT on A or the relativization of τ to A and the ITS (A, τ_A) is called an intuitionistic topological subspace of (X, τ)."

The four definitions of intuitionistic regular topological spaces are given in Yaseen & Mohammad (2012) as follows:

"**Definition 4** (Yaseen & Mohammad, 2012)**:** Suppose (X, τ) is an ITS. Then (X, τ) is said to be
(a) $R(i)$ if for each $x \in X$, F is an intuitionistic closed set of X and

$x_I \notin F$ and then there exists $U, V \in \tau$ such that $x_I \in U, F \subseteq V$ and $U \cap V = \phi_I$.

(b) $R(ii)$ if for each $x \in X$, F is an intuitionistic closed set of X and $x_{IV} \notin F$ and then there exists $U, V \in \tau$ such that $x_{IV} \in U, F \subseteq V$ and $U \cap V = \phi_I$.

(c) $R(iii)$ if for each $x \in X$, F is an intuitionistic closed set of X and $x_I \notin F$ and then there exists $U, V \in \tau$ such that $x_I \in U, F \subseteq V$ and $U \subseteq V^c$.

(d) $R(iv)$ if for each $x \in X$, F is an intuitionistic closed set of X and $x_{IV} \notin F$ and then there exists $U, V \in \tau$ such that $x_{IV} \in U, F \subseteq V$ and $U \subseteq V^c$."

Two practical examples of intuitionistic sets are given in Examples 5 and 6.

Example 5: $\mathrm{X} =$ {x|x is an inhabitant of Bangladesh}.
Define $\mathrm{A} = (\mathrm{A_T}, \mathrm{A_F})$, where
$\mathrm{A_T} =$ {x|x is an inhabitant of Bangladesh who support Argentina} and $\mathrm{A_F} =$ {y|y is an inhabitant of Bangladesh who support Brazil}.
Let us assume that the people of Bangladesh who support Argentina do not support Brazil, i.e., $A_T \cap A_F = \emptyset$. In general, $A_T \cup A_F \neq X$, since there might be some inhabitants of Bangladesh who do not support Argentina or Brazil.
In this way, we construct an intuitionistic set A of X.

Example 6: $X =$ $\{x|x$ is an inhabitant of Bangladesh$\}$. Two candidates Rahim and Karim contested an election.
$A = \{x|x$ is an inhabitant of Bangladesh who strongly support Rahim$\}$ and $B =$ $\{y|y$ is an inhabitant of Bangladesh who strongly support Karim$\}$ are subsets of X.
Supporters of Rahim= (A, B) and
Supporters of Karim= (B, A)
are intuitionistic sets of X.
Then supporters of Rahim worked on $(X - B)$ and managed p new supporters. Supporters of Karim worked on $(X - A)$ and managed q new supporters.
Let, $n(A) = a$, $n(B) = b$.
(i) If $(a + p) > (b + q)$ then Rahim will be the winner, and
(ii) if $(a + p) < (b + q)$ then Karim will be the winner.

RESULTS AND DISCUSSION

Proposition 7: Every intuitionistic subspace of an intuitionistic regular $R(i)$ space is also an intuitionistic regular $R(i)$ space. i.e., intuitionistic regular $R(i)$ is hereditary.

Proof: Suppose (X, τ) is an intuitionistic regular $R(i)$ space and (A, τ_A) is a subspace of (X, τ).

Let $a \in A$ and $E = (E_T, E_F)$ be an intuitionistic τ_A-closed set of A and $a_I \notin E$.

Therefore $E^c = (E_F, E_T)$ is an intuitionistic τ_A-open set of A.

Since (A, τ_A) is a subspace of (X, τ), then there exists an intuitionistic τ-open set $G = (G_T, G_F)$ of X such that $E^c = (E_F, E_T) = (G_T \cap A, G_F \cap A)$ implies that $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Since $a_I \notin E$, then $a \notin E_T$ implies $a \notin G_F \cap A$, which implies that $a \notin G_F$ since $a \in A$.

Hence $a_I \notin G^c$

Therefore $a \in X$ and G^c is an intuitionistic τ-closed set of X and $a_I \notin G^c$. By the definition of intuitionistic regular $R(i)$ space, there exist $U, V \in \tau$ such that $a_I \in U, G^c \subseteq V$ and $U \cap V = \phi_I$.

Since $a_I \in U$ and $a \in A$ implies that $a_I \in (U_T \cap A, U_F \cap A) = M$, which is an intuitionistic τ_A-open set of A.

Again, since $G^c \subseteq V \Rightarrow (G_F, G_T) \subseteq (V_T, V_F) \Rightarrow (G_F \cap A, G_T \cap A) \subseteq (V_T \cap A, V_F \cap A) = N$, which is an intuitionistic τ_A-open set of A. Therefore, $E \subseteq N$, since $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Now, $M \cap N = (U_T \cap A, U_F \cap A) \cap (V_T \cap A, V_F \cap A) = \big((U_T \cap A) \cap (V_T \cap A), (U_F \cap A) \cup (V_F \cap A)\big) = \big((U_T \cap V_T) \cap A, (U_F \cup V_F) \cap A\big) = (\emptyset \cap A, X \cap A) = (\emptyset, A) = \phi_{IA}$, since $U \cap V = \phi_I \Rightarrow \big((U_T \cap V_T), (U_F \cup V_F)\big) = (\emptyset, X)$. Thus (A, τ_A) is an intuitionistic regular $R(i)$ space. ∎

Proposition 8: Every intuitionistic subspace of an intuitionistic regular $R(ii)$ space is also an intuitionistic regular $R(ii)$ space. i.e., intuitionistic regular $R(ii)$ is hereditary.

Proof: Suppose (X, τ) is an intuitionistic regular $R(ii)$ space and (A, τ_A) is a subspace of (X, τ).

Let $a \in A$ and $E = (E_T, E_F)$ be an intuitionistic τ_A-closed set of A and $a_{IV} \notin E$. Therefore $E^c = (E_F, E_T)$ is an intuitionistic τ_A-open set of A.

Since (A, τ_A) is a subspace of (X, τ), then there exists an intuitionistic τ-open set $G = (G_T, G_F)$ of X such that $E^c = (E_F, E_T) = (G_T \cap A, G_F \cap A)$ implies that $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Since $a_{IV} \notin E$, then $a \in E_F = G_T \cap A$ implies $a \in G_T$. Again since $a \in G_T$ and $G^c = (G_F, G_T)$.

Hence $a_{IV} \notin G^c$.

Therefore $a \in X$ and G^c is an intuitionistic τ-closed set of X and $a_{IV} \notin G^c$. By the definition of intuitionistic regular $R(ii)$ space, there exist $U, V \in \tau$ such that $a_{IV} \in U, G^c \subseteq V$ and $U \cap V = \phi_I$.

Since $a_{IV} \in U$ and $a \in A$ implies that $a_{IV} \in (U_T \cap A, U_F \cap A) = M$, which is an intuitionistic τ_A-open set of A.

Again, since $G^c \subseteq V \Rightarrow (G_F, G_T) \subseteq (V_T, V_F) \Rightarrow (G_F \cap A, G_T \cap A) \subseteq (V_T \cap A, V_F \cap A) = N$, which is an intuitionistic τ_A-open set of A. Therefore, $E \subseteq N$, since $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Now, $M \cap N = (U_T \cap A, U_F \cap A) \cap (V_T \cap A, V_F \cap A) = \big((U_T \cap A) \cap (V_T \cap A), (U_F \cap A) \cup (V_F \cap A)\big) = \big((U_T \cap V_T) \cap A, (U_F \cup V_F) \cap A\big) = (\emptyset \cap A, X \cap A) = (\emptyset, A) = \phi_{IA}$, since $U \cap V = \phi_I \Rightarrow \big((U_T \cap V_T), (U_F \cup V_F)\big) = (\emptyset, X)$. Thus (A, τ_A) is an intuitionistic regular $R(ii)$ space. ∎

Proposition 9: Every intuitionistic subspace of an intuitionistic regular $R(iii)$ space is also an intuitionistic regular $R(iii)$ space. i.e., intuitionistic regular $R(iii)$ is hereditary.

Proof: Suppose (X, τ) is an intuitionistic regular $R(iii)$ space and (A, τ_A) is a subspace of (X, τ).

Let $a \in A$ and $E = (E_T, E_F)$ be an intuitionistic τ_A-closed set of A and $a_I \notin E$. Therefore $E^c = (E_F, E_T)$ is an intuitionistic τ_A-open set of A.

Since (A, τ_A) is a subspace of (X, τ), then there exists an intuitionistic τ-open set $G = (G_T, G_F)$ of X such that $E^c = (E_F, E_T) = (G_T \cap A, G_F \cap A)$ implies that $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Since $a_I \notin E$, then $a \notin E_T$ implies $a \notin G_F \cap A$, which implies that $a \notin G_F$ since $a \in A$.

Hence $a_I \notin G^c$.

Therefore $a \in X$ and G^c is an intuitionistic τ-closed set of X and $a_I \notin G^c$. By the definition of intuitionistic regular $R(iii)$ space, there exist $U, V \in \tau$ such that $a_I \in U, G^c \subseteq V$ and $U \subseteq V^c$.

Since $a_I \in U$ and $a \in A$ implies that $a_I \in (U_T \cap A, U_F \cap A) = M$, which is an intuitionistic τ_A-open set of A.

Again, since $G^c \subseteq V \Rightarrow (G_F, G_T) \subseteq (V_T, V_F) \Rightarrow (G_F \cap A, G_T \cap A) \subseteq (V_T \cap A, V_F \cap A) = N$, which is an intuitionistic τ_A-open set of A. Therefore, $E \subseteq N$, since $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Since, $U \subseteq V^c \Rightarrow (U_T,\ U_F) \subseteq (V_F, V_T) \Rightarrow (U_T \cap A,\ U_F \cap A) \subseteq (V_F \cap A, V_T \cap A) \Rightarrow M \subseteq N^c$.

Thus (A, τ_A) is an intuitionistic regular $R(iii)$ space. ■

Proposition 10: Every intuitionistic subspace of an intuitionistic regular $R(iv)$ space is also an intuitionistic regular $R(iv)$ space. i.e., intuitionistic regular $R(iv)$ is hereditary.

Proof: Suppose (X, τ) is an intuitionistic regular $R(iv)$ space and (A, τ_A) is a subspace of (X, τ).

Let $a \in A$ and $E = (E_T, E_F)$ be an intuitionistic τ_A-closed set of A and $a_{IV} \notin E$. Therefore $E^c = (E_F, E_T)$ is an intuitionistic τ_A-open set of A.

Since (A, τ_A) is a subspace of (X, τ), then there exists an intuitionistic τ-open set $G = (G_T, G_F)$ of X such that $E^c = (E_F, E_T) = (G_T \cap A, G_F \cap A)$ implies that $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Since $a_{IV} \notin E$, then $a \in E_F = G_T \cap A$ implies $a \in G_T$. Again since $a \in G_T$ and $G^c = (G_F, G_T)$.

Hence $a_{IV} \notin G^c$.

Therefore $a \in X$ and G^c is an intuitionistic τ-closed set of X and $a_{IV} \notin G^c$. By the definition of intuitionistic regular $R(iv)$ space, there exist $U, V \in \tau$ such that $a_{IV} \in U, G^c \subseteq V$ and $U \subseteq V^c$.

Since $a_{IV} \in U$ and $a \in A$ implies that $a_{IV} \in (U_T \cap A, U_F \cap A) = M$, which is an intuitionistic τ_A-open set of A.

Again, since $G^c \subseteq V \Rightarrow (G_F, G_T) \subseteq (V_T, V_F) \Rightarrow (G_F \cap A, G_T \cap A) \subseteq (V_T \cap A, V_F \cap A) = N$, which is an intuitionistic τ_A-open set of A. Therefore, $E \subseteq N$, since $E = (E_T, E_F) = (G_F \cap A, G_T \cap A)$.

Since, $U \subseteq V^c \Rightarrow (U_T,\ U_F) \subseteq (V_F, V_T) \Rightarrow (U_T \cap A,\ U_F \cap A) \subseteq (V_F \cap A, V_T \cap A) \Rightarrow M \subseteq N^c$.

Thus (A, τ_A) is an intuitionistic regular $R(iv)$ space. ■

CONCLUSIONS AND SUGGESTIONS

In conclusion, this paper's primary findings, encapsulated within Propositions 7 through 10, solidify the assertion that the intuitionistic regular spaces $R(i)$, $R(ii)$, $R(iii)$, and $R(iv)$ inherently possess the hereditary property. This discovery contributes significantly to the foundational understanding of intuitionistic topological spaces. By firmly establishing the hereditary nature of these spaces, we have laid a critical cornerstone for future investigations in this field. These results not only enhance our comprehension of intuitionistic topological spaces but also pave the way for more nuanced and advanced explorations, which may uncover deeper insights into their structural characteristics and broader applications.

As we move forward, we recommend that future research endeavors build upon these established findings. Researchers are encouraged to delve deeper into the implications and ramifications of the hereditary property in intuitionistic regular spaces, exploring its connections with other mathematical concepts and its potential applications in various scientific and practical domains. Additionally, the insights garnered from this study could serve as a springboard for the development of novel methodologies and problem-solving approaches within the realm of intuitionistic topological spaces, further advancing our understanding and harnessing the potential of these spaces in diverse contexts.

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