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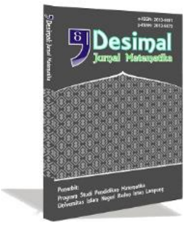

A review of some properties of persistent homology

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ABSTRACT

Every day, enormous complex geometric data are accumulating rapidly, and qualitative analysis is needed, which cannot be done properly without studying the shapes of those data. Persistent homology describes the homology of data sets of arbitrary size, producing state-of-the art results in data analysis across a significant number of fields and sparking a rigorous study of persistence in homology theory. In this study, persistent homology has been demonstrated as a homology theory by satisfying the Eilenberg-Steenrod axioms. A brief background on persistent homology groups has been written to understand their construction. Then other definitions of persistent homology based on functors and graded modules have also been reviewed. The Mayer-Vietoris-Vietorisfor persistent homology has been derived as a property of persistent homology. Subsequently, a long, exact sequence for persistent homology has been constructed. Furthermore, the stability of persistent homology has been examined carefully. Finally, the Diamond principle of persistent homology has been explained briefly. This study can be used to investigate new properties of persistent homology, among other benefits.

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INTRODUCTION

The continuous advancement of information and technology leads to the constant accumulation of diverse data types. This influx has far-reaching effects across sectors like research, business, and healthcare, emphasizing the importance of utilizing this wealth of information for informed decision-making and valuable insights. Among them, some data can be analyzed using statistical or numerical methodologies. But many data sets contain complex geometric components like images, signals, networks, etc. Recent research reveals that some methods of algebraic topology are effective in dealing with complicated data. This kind of research is known as Topological Data Analysis (TDA), and a common mathematical tool in TDA is persistent homology, which is an algebraic method to determine topological features or a data set by using filtration, which comes from a suitable function on it.

Persistent homology is a tool for computing a topological feature called the shape of a structure. In the past few decades, persistent homology has emerged as one of the central points of the modern computational topology study field. Nowadays, persistent homology is a wide-ranging and attractive topic for mathematicians, computer scientists, data analysts, and scientists. It is important to theoretically have good control over persistent homology by gaining good knowledge of persistent homology and its properties.

Persistent homology has been utilized to evaluate complicated geometric data in the fields of biology and chemistry, astrophysics, automatic image classification, sensor analysis, and social network analysis (Fugacci, Scaramuccia, Iuricich, & de Floriani, 2016). Persistence diagrams were used to discover a subgroup of breast cancer in 2010 (Nicolau, Levine, & Carlsson, 2011). The state-of-the-art in assessing biological nanostructures was reached using persistence-based clustering (Pike et al., 2020). Additionally, persistent homology has been discovered in a variety of data analysis fields, such as neuroscience (Sizemore, Phillips-Cremins, Ghrist, & Bassett, 2019), time series data (Seversky, Davis, & Berger, 2016), and shape analysis (Gamble & Heo, 2010), among others. In order to summarize the form of data across various scales and deliver reliable findings for noisy data, persistent homology is essential. Due to these two distinctive qualities, persistent homology was given great consideration for investigating its other aspects. The literature is also very scarce because the emergence of persistent homology is a recent phenomenon that is continually evolving (Fugacci et al., 2016). Thus, there is a need for greater review writing. The majority of current research on persistent homology—such as that by Graff et al. (2021), Aktas, Akbas, & Fatmaoui (2019), and Meng, Anand, Lu, Wu, & Xia (2020)—focuses on its applications in many domains. Due to the wide range of users of this instrument, some review studies (Fugacci et al., 2016; Koplik, 2019) have been conducted to facilitate their use. Gunnar Carlsson outlined various persistent homology research directions in Carlsson (2020), which might be quite useful for examining persistent homology's essential elements. Only the Mayer-Vietoris formula and Long (precise) sequences for persistent homology were discussed among the homological qualities in Varli, Yilmaz, & Pamuk (2018). Regarding numerous linkages to the applications in Edelsbrunner & Harer (2008), persistent homology had been examined. The Eilenberg-Steenrod axioms, the Mayer-Vietoris formula, the Long exact sequence, stability, the Diamond principle, and its connections to other homology theories and other algebraic topological structures are all properties of persistent homology that require further study.

METHOD

In this study, simplex, simplicial complex, and simplicial homology have been reviewed. Using these Eilenberg-Steenrod Axioms, persistent homology has been checked to see whether it is a homology theory or not. Finally, some other properties of persistent homology have been reviewed briefly, as mentioned.

Definition 1 (Munkres, 2018)**:**
A $m-simplex$ σ is a convex hull of $m+1$ geometrically independent points $v_0, v_1, \cdots, v_m \in \mathbb{R}^n$ are denoted by $\langle v_0, v_1, \cdots, v_m \rangle$, is the set of all points $x \in \mathbb{R}^n$ of the form

$$x = \sum_{i=0}^{m} t_i v_i$$

where

$$\sum_{i=0}^{m} t_i = 1$$

and each $t_i \geq 0$.

The $m+1$ points are called the *vertices* and m is called the *dimension* of the simplex.

Example 1:

(i) A $0-$*simplex* $\langle a_0 \rangle$ is just a point in $\mathbb{R}^n$.

Figure 1. 0 $-$simplex

(ii) A $1-$*simplex* $\langle a_0, a_1 \rangle$ is the line segment joining the points a_0 and a_1 geometrically.

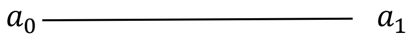


Figure 2. 1 $-$simplex

(iii) A $2-$*simplex* $\langle a_0, a_1, a_2 \rangle$ is a triangle with the vertices a_0, a_1 and a_2.

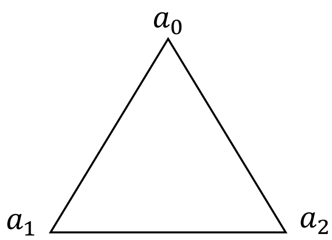


Figure 3. 2 $-$simplex

(iv) A $3-$*simplex* is a tetrahedron having four vertices.

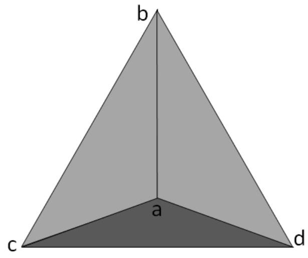


Figure 4. 3 $-$simplex

Definition 2 (Munkres, 2018)**:**

A *simplicial complex* Δ is a collection of simplices such that

a) If Δ contains a simplex σ, then it contains every face of σ.
b) If two simplices in Δ intersects, then their intersection is either empty or a face of each of them.

The *dimension* of a simplicial complex Δ is the dimension of the largest face of Δ.

A simplicial complex is said to be *finite* if its vertex set is finite.

Definition 3:

Let σ be a simplex. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If dim $\sigma > 0$, the orderings of the vertices of σ then fall into two equivalent classes; each of these classes is called an orientation of σ. (If the dim $\sigma = 0$, then there is only one class and hence only one orientation of σ). An *oriented simplicial complex* is a simplicial complex together with the orientation of all its simplices. In further parts of this article, square brackets will be used instead of $\langle \cdot \rangle$ to denote an oriented simplicial complex.

Example 2:

Let $\sigma = [v_0, v_1, v_2, v_3]$, the tetrahedron of vertices v_0, v_1, v_2 and v_3. Consider the two orderings of the vertices v_0, v_1, v_2, v_3 and v_0, v_3, v_1, v_2. $(v_1, v_3, v_2) = (v_1, v_2) \circ (v_1, v_3)$, this is an even permutation; therefore, the orderings are equivalent.

Definition 4:

Let Δ be an oriented simplicial complex, and let R be a ring. Define $C_i(\Delta, R)$ as the free $R-$module with basis all the oriented $i-$simplices, modulo the relations

$$[v_0, v_1, \cdots, v_i] \sim (-1)^{sgn(\alpha)} \left[v_{\alpha_{(0)}}, v_{\alpha_{(1)}}, \cdots, v_{\alpha_{(i)}}\right]$$

where α is a permutation made up of the two orderings $v_0, v_1, \cdots, v_i$ and $v_{\alpha_{(0)}}, v_{\alpha_{(1)}}, \cdots, v_{\alpha_{(i)}}$. $sgn(\alpha)$ is zero if α is an even permutation; otherwise, it is one.

Example 3:
Let us consider the simplicial complex
$\Delta = \{[v_0, v_1, v_2], [v_0, v_1], [v_0, v_2], [v_1, v_2], [v_2, v_3], [v_2, v_4], [v_2, v_5], [v_3, v_4]\}$.
Then the bases for $C_0(\Delta, R)$, $C_1(\Delta, R)$ and $C_2(\Delta, R)$ are respectively

$\{[v_0], [v_1], [v_2], [v_1, v_2], [v_3], [v_4], [v_5]\}$,

$\left\{\begin{matrix}[v_0, v_1], [v_0, v_2], [v_1, v_2], [v_2, v_3], \\ [v_2, v_4], [v_2, v_5], [v_3, v_4]\end{matrix}\right\}$,

and $\{[v_0, v_1, v_2]\}$.

And all $C_i(\Delta, R) = \{0\}$ for $i > 2$.

Definition 5:
Let Δ be a finite simiplicial complex. For $n \geq 1$, define

$\partial_n : C_n(\Delta, R) \rightarrow C_{n-1}(\Delta, R)$ by

$$\partial_n([v_0, v_1, \cdots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots, \hat{v}_i, \cdots, v_n]$$

The notation $\hat{v}_i$ means that the vertex v_i is omitted. Also, for $n = 0$, define $\partial_0 : C_0(\Delta, R) \rightarrow C_{-1}(\Delta, R)$ to be identically zero (as $C_{-1}(\Delta, R) = 0$). The map ∂_n is often called the *simplicial boundary map* of nth order.

Proposition 1: For all $n \geq 0$, $\partial_{n-1}\partial_n = 0$.

Definition 6 (Munkres, 2018):
For each $n \geq 0$, ∂_n is a $R-$module homomorphism. The sub-module $Ker\ \partial_n \subseteq C_n(\triangle, R)$ is denoted by $Z_n(\triangle, R)$; its elements are called *simplicial $n-$cycles*.
The sub-module $Im\partial_{n+1} \subseteq C_n(\triangle, R)$ is denoted by $B_n(\triangle, R)$; its elements are called *simplicial n-boundaries*.
The n-th *simplicial homology* module is defined by

$$H_n(\triangle, R) = \frac{Z_n(\triangle, R)}{B_n(\triangle, R)}$$

Example 4:
Simplicial homology for a sphere.
One can triangulate a sphere as a tetrahedron.
Suppose Δ be the simplicial complex defined by the boundary of a tetrahedron, having maximal faces
$\{[a, b, c], [a, b, d], [a, c, d], [b, c, d]\}$.

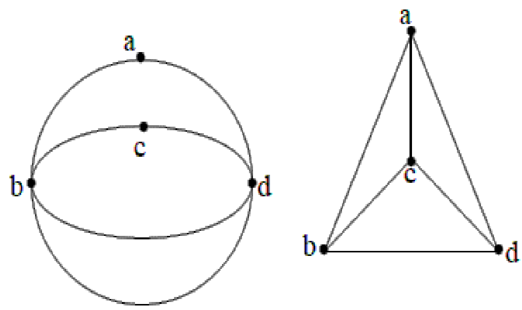


Figure 5. Triangulation of Sphere

The bases for $C_0(\triangle, R)$, $C_1(\triangle, R)$ and $C_2(\triangle, R)$ are respectively
$\{[a], [b], [c], [d]\}$,
$\{[a, b], [a, c], [a, d], [b, c], [b, d], [c, d]\}$ and
$\{[a, b, c], [a, b, d], [a, c, d], [b, c, d]\}$.
Here $C_i(\triangle, R) = 0$ for $i \neq 0, 1$ and 2.

The abstract simplicial complex $C_*(\triangle, R)$ is given by

$$0 \xrightarrow{\partial_3} R^4 \xrightarrow{\partial_2} R^6 \xrightarrow{\partial_1} R^4 \xrightarrow{\partial_0} 0$$

Here, ∂_0 is the zero map, i.e., the $\ker\partial_0 = R^4$, ∂_3 is trivial, ∂_1 is defined by

$$\partial_1([a, b]) = [b] - [a] = (-1) \cdot [a] + 1 \cdot [b] + 0 \cdot [c] + 0 \cdot [d].$$

$\partial_1([a,c]) = [c] - [a]$
$= (-1)\cdot[a] + 0\cdot[b] + 1\cdot[c] + 0\cdot[d]$
$\partial_1([a,d]) = [d] - [a]$
$= (-1)\cdot[a] + 0\cdot[b] + 0\cdot[c] + 1\cdot[d]$
$\partial_1([b,c]) = [c] - [b]$
$= 0\cdot[a] + (-1)\cdot[b] + 1\cdot[c] + 0\cdot[d]$
$\partial_1([b,d]) = [d] - [b]$
$= 0\cdot[a] + (-1)\cdot[b] + 0\cdot[c] + 1\cdot[d]$
$\partial_1([c,d]) = [d] - [c] = 0\cdot[a] + 0\cdot[b] + (-1)\cdot[c] + 1\cdot[d]$.

The matrix representation of $\partial_1$is

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

by transposing we have

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence $\{(-1,1,0,0),(0,1,-1,0),(0,0,1,-1)\}$ is a basis for $Im\partial_1$, therefore the dimension of $Im\partial_1$ is 3, consequently we have the dimension of $ker\partial_1$ is 3.

Thus,

$$H_0(\triangle, R) = \frac{ker\partial_0}{Im\partial_1} \sim\ <(0,0,0,1)> \sim R^1$$

∂_2 is defined by

$\partial_2([a,b,c]) = [b,c] - [a,c] + [a,b]$
$= 1\cdot[a,b] + (-1)\cdot[a,c] + 0\cdot[a,d] + 1\cdot[b,c] + 0\cdot[b,d] + 0\cdot[c,d]$
$\partial_2([a,b,d]) = [b,d] - [a,d] + [a,b]$
$= 1\cdot[a,b] + 0\cdot[a,c] + (-1)\cdot[a,d] + 0\cdot[b,c] + 1\cdot[b,d] + 0\cdot[c,d]$
$\partial_2([a,c,d]) = [c,d] - [a,d] + [a,c]$
$= 0\cdot[a,b] + 1\cdot[a,c] + (-1)\cdot[a,d] + 0\cdot[b,c] + 0\cdot[b,d] + 1\cdot[c,d]$
$\partial_2([b,c,d]) = [c,d] - [b,d] + [b,c]$
$= 0\cdot[a,b] + 0\cdot[a,c] + 0\cdot[a,d] + 1\cdot[b,c] + (-1)\cdot[b,d] + 1\cdot[c,d]$

The matrix representation of $\partial_2$is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

by transposing we have

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\{(1,-1,0,1,0,0),(0,1,-1,-1,1,0),(0,0,0,1,-1,1)\}$ is a basis for $Im\partial_2$.

$\partial_1\big((1,-1,0,1,0,0)\big) = \partial_1\big((0,1,-1,-1,1,0)\big) = \partial_1\big((0,0,0,1,-1,1)\big) = 0$ implies that

$$ker\partial_1 = Im\partial_2$$

Thus,

$$H_1(\triangle, R) = \frac{ker\partial_1}{Im\partial_2} = 0$$

Suppose $\partial_2(x,y,z,u) = 0$, then we have the system of equations

$$\begin{matrix} x+y=0 \\ -x+z=0 \\ -y-z=0 \\ x+u=0 \\ y-u=0 \\ z+u=0 \end{matrix} \sim \begin{matrix} x+y=0 \\ y+z=0 \\ z+u=0 \end{matrix}$$

Here, u is the only one free variable, if we put $u=1$, we have $z=-1, y=1$ and $x=-1$. Therefore, $ker\partial_2$ has a basis $\{(-1,1,-1,1)\}$.

Thus,

$$H_2(\triangle, R) = \frac{ker\partial_2}{Im\partial_3} \sim \; < (-1,1,-1,1) > \sim R^1$$

and all $H_i(\triangle, R) = 0$ for $i \neq 0, 1$ and 2. Hence $H_0(\triangle, R)$ has only one basis element i.e., the simplicial complex is connected.

Definition 7:
Let $f: K \to L$ be a map between two simplical complexes. Then for each dimension p, the homomorphism induced by f is defined as

$$f_*: H_p(K) \to H_p(L).$$

Definition 8:
A *filtration* of a simplicial complex X is a nested sequence of complexes
$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$, where X_i is the sets of i −simplices; $i = 0,1,\cdots,n$. We call X a filtered complex.

Definition 9 (Zomorodian & Carlsson, 2005)**:**
Consider a filtrated simplicial complex

$$K^0 \subseteq K^1 \subseteq K^2 \subseteq K^3 \subseteq \cdots$$

corresponding to each filtration index I, there are associated free modules C_p^a over a principle ideal domain (PID) R, boundary operators $\partial_p^a, B_p^a, Z_p^a$ and homology H_p^a for all $p \geq 0$.
The *b-persistent* p *−th homology group* (Zomorodian & Carlsson, 2005) of K^a is

$$H_p^{a,b} = \frac{Z_p^a}{B_p^{a+b} \cap Z_p^a}, \forall\, a < b \in \mathbb{N}$$

Since B_p^{a+b} and Z_p^a are submodules of C_p^{a+b}, their intersection is also a submodule, so the definition is well defined.
We also define persistent homology using an inclusion map.
(Varli et al., 2018) Given a topological space X, let $f: X \to \mathbb{R}$ be a continuous function. Then a sublevel set of X is defined as

$$X_a = \{x \in X: f(x) \leq a\}, \text{for all } a \in \mathbb{R}.$$

Note that $\mathcal{F} = \{X_a: a = 1,2,\cdots,n\}$ gives a filtration of the space X.
For $a < b \in \mathbb{N}$, the p −th persistent homology group $H_p^{a,b}(X)$ is defined as the image of the homomorphism $j_p^{a,b}: H_p(X_a) \to H_p(X_b)$ induced by the inclusion $i: X_a \to X_b$,

$$H_p^{a,b}(X) := Im\, j_p^{a,b}.$$

In other words, we say that the persistent homology groups consist of the homology classes of X_a that are still alive at X_b.

Definition 10:
A *persistence complex* $\mathfrak{C}$ is a family of chain complexes C_*^a over R, together with chain maps $f^a: C_*^a \to C_*^{a+1}\ \forall\, a \geq 0$, so the following diagram is commutative.

$$\begin{array}{ccccccccc}
\cdots \to & C_2^0 & \xrightarrow{\partial_2^0} & C_1^0 & \xrightarrow{\partial_1^0} & C_0^0 & \to 0 \\
& \downarrow f_2^0 & & \downarrow f_1^0 & & \downarrow f_0^0 & \\
\cdots \to & C_2^1 & \xrightarrow{\partial_2^1} & C_1^1 & \xrightarrow{\partial_1^1} & C_0^1 & \to 0 \\
& \downarrow f_2^1 & & \downarrow f_1^1 & & \downarrow f_0^1 & \\
\cdots \to & C_2^2 & \xrightarrow{\partial_2^2} & C_1^2 & \xrightarrow{\partial_1^2} & C_0^2 & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

The filtration index decreases vertically to the top under chain maps f^a and the dimension of chain complexes decreases horizontally to the right.

Example 5:
Given a finite set of points X from a subspace $x \in \mathbb{R}^n$. We call X point cloud data (PCD). We obtain a persistence complex using Rips complexes. We get a persistent homology group from point cloud data.

Definition 11:
A *persistence module* $\mathcal{H}$ is a family of the homology H^a of a persistence complex, together with the induced homomorphism $\alpha^a: H^a \to H^{a+1}$ of the inclusion maps $f^a: C^a \to C^{a+1}$. A persistence module is said to be finite if each component of the module is a finitely generated R −module and if the maps α^a are isomorphisms for $a \geq m$, for some positive integer m.

RESULTS AND DISCUSSION

1. PERSISTENCE MODULE AS FUNCTOR AND GRADED MODULE

Definition 12:
Let F be a field, and the category of finite-dimensional vector spaces over F be denoted by V^F. Let $(\mathbb{R}, \geq)$ be a posetal category. Then a *persistence module* is a functor $H: (\mathbb{R}, \geq) \to V^F$. So, it is given by a family $\{H_a\}_{a\in\mathbb{R}}$ of finite-dimensional vector spaces over F together with morphisms $\varphi: F_a \to F_b$ for all $a \leq b$ such that Figure 6 is commutative.

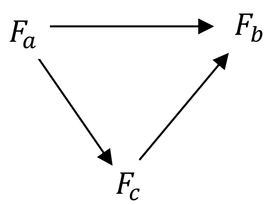


Figure 6. Diagram of Morphisms among $\boldsymbol{F_a}, \boldsymbol{F_b}$ and $\boldsymbol{F_c}$.

A morphism of persistence modules is just a natural transformation.

Definition 13 (Varli et al., 2018):
Let X be a topological space, and $\emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_a \subset \cdots \subset X_n = X$ be a filtration of X. The p −*persistence module* X is a family of k −homology modules $H_p^a(X)$, together with module homomorphisms $i_p^a: H_p^a(X) \to H_p^{a+1}(X)$ induced by the inclusion maps $i_a: X_a \to X_{a+1}$.

A persistence module is *finite* if each component module is finitely generated. The p −persistence module can be given by the structure of a *finitely generated graded module* over a polynomial ring $R[x]$,

$$\mathcal{H}_p(X) = \oplus_{a=0}^{n} H_p^a(X)$$

which is defined as

$$x \cdot (m^0, m^1, \cdots, m^n) = \left(0, i_p^0(m^0), i_p^1(m^1), \cdots, i_p^n(m^n)\right),$$

where $i_p^n(m^n) = m^n$, for $m^n \in H_p^n(X)$.

Theorem 1:
The persistence module of the graded module over PID R decomposes uniquely into the form

$$\mathcal{H}_p(X) = (\oplus_{i=1}^{n} \Sigma^{\alpha_i} R) \oplus \left(\oplus_{j=1}^{m} \Sigma^{\beta_j} R / \left(t_j R\right)\right)$$

where $t_j \in R$ are such that $t_j | t_{j+1}, \alpha_i, \beta_j \in \mathbb{Z}$ and Σ^α represents an α −shift upward grading.

Proof: Since R is a principle ideal domain, then we obtain

$$\mathcal{H}_p(X) = (\oplus_{i=1}^{n} \Sigma^{\alpha_i} R) \oplus \left(\oplus_{j=1}^{m} \Sigma^{\beta_j} R / \left(t_j R\right)\right)$$

from Chazal, Silva, Glisse, & Oudot (2016). The proof is complete.

2. EILENBERG-STEENROD AXIOMS

Definition 14:

Let X be a topological space and A be a subspace of X. A sequence of functors H_p from the category of pairs (X, A) of topological spaces to the category of abelian groups satisfies the *Eilenberg-Steenrod Axioms* if the following conditions hold:

1. g_* is identity isomorphism if g is identity homeomorphism,
where $g: (X, A) \to (Y, B)$
and $g_*: H_p(X, A) \to H_p(Y, B)$.

$1. A. (hg)_* = h_* g_*$, where $h: (Y, B) \to (Z, C)$.

$1. B.$ If $(X, A) = \coprod_\beta (X_\beta, A_\beta)$, the disjoint union of pairs (X_β, A_β), then $H_p(X, A) \cong \oplus_p (X_\beta, A_\beta)$.

2. **Homotopy Axiom**: If $g: (X, A) \to (Y, B)$ is homotopic to $f: (X, A) \to (Y, B)$, then their induction maps $g_*: H_p(X, A) \to H_p(Y, B)$ and $f_*: H_p(X, A) \to H_p(Y, B)$, are same in the category.

3. **Exactness Axiom**: For all pairs (X, A), there is a homology sequence

$$\cdots \to H_{p+1}(X, A) \xrightarrow{\partial} H_p(A) \xrightarrow{k_*} H_p(X) \xrightarrow{l_*} H_p(X, A) \xrightarrow{\partial} H_{p-1}(A) \to \cdots$$

which is exact, where $k: A \to X$ and $l: (X, \emptyset) \to (X, A)$ are inclusion maps which induce maps $k_*: H_p(A) \to H_p(X)$ and $l_*: H_p(X) \to H_p(X, A)$. The map $\partial: H_p(X, A) \to H_{p-1}(A)$ is called the boundary map.

4. **Excision Axiom**: If (X, A) is a pair of space and $U \subseteq A$ a subset with $\overline{U} \subseteq int\ A$, then the inclusion map of pairs $i: (X - U, A - U) \to (X, A)$ induces an isomorphism $i_*: H_p(X - U, A - U) \to H_p(X, A)$.

5. **Dimension Axiom**: Let a be a single point space. Then $H_p(a) = 0$ when $p \neq 0$, in which case $H_0(a) = G$, where G are some groups. The group $H_0(a)$ is called the coefficient group.

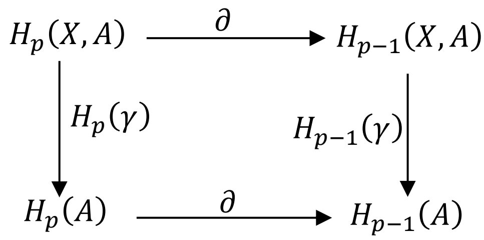


Figure 7. Diagram of sequence of functors $\boldsymbol{H_p}$

These axioms generalize a homology theory. If the dimension axiom is omitted, then the remaining axioms are called an extraordinary homology theory. In this way, we generalize a co-homology theory where H_p (see Figure 7) is required to be a co-functor (meaning the induced map points in the opposite direction).

3. IS PERSISTENT HOMOLOGY A HOMOLOGY THEORY?

We know that homology theory satisfies the Eilenberg-Steenrod Axioms. Here, we showed that persistent homology holds Eilenberg-Steenrod Axioms. The exactness axiom was proved in Varli et al. (2018), and the excision axiom was proved in Palser (2019). These two articles have been reviewed, and we have added the review here. The homotopy axiom and the dimension axiom of Eilenberg-Steenrod Axioms for persistent homology have been proved below.

1. Homotopy Axiom: If $f \simeq g: (X, U) \to (Y, V)$, then their induction maps $H_p^{a,b}(f) = H_p^{a,b}(g): H_p^{a,b}(X, U) \to H_p^{a,b}(Y, V)$.

Proof: Since $H_p^{a,b}(f)$ is a restriction map of f_* and $H_p^{a,b}(g)$ is a restriction map of g_* and
$f_* = g_*: H_p^b(X, U) \to H_p^b(Y, V)$,
Then we get,
$H_p^{a,b}(f) = H_p^{a,b}(g)$. (Dawson, 1988)
Where, $H_p^b(X, U)$ and $H_p^b(Y, V)$ are p −th simplicial homology of the pairs at b −filtration step, and f_*, g_* are inductive homomorphism from
$H_p^b(X, U) \to H_p^b(Y, V)$.
The proof is complete.

2. Exactness Axiom: The long sequence for the graded persistence module of the pair (X, U) is exact. For any $a < b \in \mathbb{N}$, we get Figure 8.

$$\begin{array}{ccccccc} \cdots \to & H_{p+1}(X_a, U_a) & \xrightarrow{\delta_p^a} & H_p(U_a) & \xrightarrow{\alpha_p^a} & H_p(X_a) \\ & \downarrow \gamma_{p+1} & & \downarrow \varphi_p & & \downarrow \omega_p \\ \cdots \to & H_{p+1}(X_b, U_b) & \xrightarrow{\delta_p^b} & H_p(U_b) & \xrightarrow{\alpha_p^b} & H_p(X_b) \end{array}$$

$$\begin{array}{cccccc} \xrightarrow{\beta_p^a} & H_p(X_a, U_a) & \xrightarrow{\delta_p^a} & H_{p-1}(A_a) & \to \cdots \\ & \downarrow \gamma_p & & \downarrow \varphi_{p-1} & \\ \xrightarrow{\beta_p^b} & H_p(X_b, U_b) & \xrightarrow{\delta_p^b} & H_{p-1}(U_b) & \to \cdots \end{array}$$

Figure 8. Induced homomorphisms by inclusion and quotient maps

Where $\delta_p^a([c]) = \left[(\partial c)|_{U_a}\right]$ and α_p^a, β_p^a are homomorphisms induced by inclusion and quotient maps, respectively.
(Varli et al., 2018) The long sequence of the graded persistence modules of pair (X, U)

$$\cdots \to \mathcal{H}_{p+1}(X, U) \xrightarrow{\delta} \mathcal{H}_p(U) \xrightarrow{\alpha} \mathcal{H}_p(X) \xrightarrow{\beta} \mathcal{H}_p(X, U) \to \cdots$$

is exact, where $\delta(n^0, n^1, \cdots, n^m) = \left(\delta_p^0(n^0), \delta_p^1(n^1), \cdots, \delta_p^m(n^m)\right)$,

$$\alpha(y^0, y^1, \cdots, y^m) = \left(\alpha_p^0(y^0), \alpha_p^1(y^1), \cdots, \alpha_p^m(y^m)\right),$$

$$\beta(x^0, x^1, \cdots, x^m) = \left(\beta_p^0(x^0), \beta_p^1(x^1), \cdots, \beta_p^m(x^m)\right).$$

That is, $(i)\ Im\ \delta = Ker\ \alpha$,
$(ii)\ Im\ \alpha = Ker\ \beta$,
$(iii)\ Im\ \beta = Ker\ \delta$.
This theorem has been proved after introducing a long, exact sequence of persistent homology later.

3. Excision Axiom: Let X be a triangulable topological space with subspaces U and V such that $X = int\ U \cup int\ V$. Let $\{X_a\}$ be a filtration of X, and consider $\{U_a\}$ and $\{V_a\}$ are filtrations on U and V such that $X_a = int\ U_a \cup int\ V_a$. Then there is an isomorphism of persistence modules

$$\mathcal{H}_p(X, U) \cong \mathcal{H}_p(V, U \cap V)$$

and an isomorphism of persistent homology groups

$$H_p^{a,b}(X, U) \cong H_p^{a,b}(V, U \cap V), \forall\, p \in \mathbb{N}.$$

Proof: Filtrations of X, U, V, and $U \cap V$ give sequences of inclusions of pairs as follows:

$$(X_0, U_0) \hookrightarrow (X_1, U_1) \hookrightarrow \cdots \hookrightarrow (X, U)$$

and

$$(V_0, (U \cap V)_0) \hookrightarrow (V_1, (U \cap V)_1) \hookrightarrow \cdots \hookrightarrow (V, (U \cap V))$$

Consider the following diagram, where each map is induced by the inclusion of pairs:

$$\begin{array}{ccc} H_p(V_0, (U \cap V)_0) & \to & H_p(V_1, (U \cap V)_1) \to \\ \downarrow & & \downarrow \end{array}$$

$$H_p(X_0, U_0) \rightarrow H_p(X_1, U_1) \rightarrow \cdots$$

$$\cdots \rightarrow H_p(V, (U \cap V))$$
$$\downarrow$$
$$\cdots \rightarrow H_p(X, U)$$

As $X_a \subseteq int\, U_a \cup int\, V_a, \forall\, a \in \mathbb{N}$, every vertical map in the above diagram is an isomorphism by the usual excision theorem. Hence, we have an isomorphism of persistence modules

$$\mathcal{H}_p(X, U) \cong \mathcal{H}_p(V, U \cap V)$$

from Palser (2019).

Since $H_p(V_a, (U \cap V)_a) \cong H_p(V_b, (U \cap V)_b)$ and $H_p(X_a, U_b) \cong H_p(X_b, U_b)$, so we have

$$H_p(X_b, U_b) \cong H_p(V_b, (U \cap V)_b)$$

Thus, we get

$$H_p^{a,b}(X, U) \cong H_p^{a,b}(V, U \cap V), \forall\, p \in \mathbb{N}.$$

The proof is complete.

4. **Dimension Axiom**: Let x be a single point space. Then $H_p^{a,b}(x) = 0$ when $p \neq 0$, and
$H_0^{a,b}(x) = R$, where R is an Euclidean space.

Proof: Since $H_p^{a,b}(x) \subset H_p^b(x)$ and $H_p^b(x) = 0, \forall\, p \neq 0$, then we get

$$H_p^{a,b}(x) = 0, \forall\, p \neq 0 \text{ and } H_p^b(x) = R.$$

Where $a < b \in \mathbb{N}$. The proof is complete.

We had shown that persistent homology holds the Homotopy Axiom, Exactness Axiom, Excision Axiom, and Dimension Axiom of the Eilenberg-Steenrod Axioms. So, we say that persistent homology satisfies the Eilenberg-Steenrod Axioms. Thus, persistent homology is a homology theory.

4. MAYER-VIETORIS FORMULA FOR PERSISTENT HOMOLOGY

Definition 15:

Let X be a topological space with subspaces U and V such that $X = int\, U \cup int\, V$. Then the *Mayer-Vietoris exact sequence* has the following form

$$\cdots \rightarrow H_{p+1}(X) \xrightarrow{\delta_p} H_p(U \cap V) \xrightarrow{\varphi_p} H_p(U) \oplus H_p(V) \xrightarrow{\sigma_p} H_p(X) \rightarrow \cdots \rightarrow H_0(X) \rightarrow 0$$

where $\delta_p([x]) = [\partial(x|_U)], \varphi_p([y]) = ([y], [-y])$ and $\sigma_p([z], [\acute{z}]) = [z + \acute{z}]$.

A Mayer-Vietoris sequence would increase our ability in terms of visualization by partitioning a set of data into a number of parts that are easily recognizable.

Let X_a, U_a, V_a, and $(U \cap V)_a$ denote the sublevel sets as in Fabio & Landi (2011). Consider the following diagram:

$$\begin{array}{ccc} \rightarrow H_{p+1}(X_a) & \xrightarrow{\delta_p^a} & H_p((U \cap V)_a) \\ \downarrow \gamma_{p+1} & & \downarrow \alpha_p \\ \cdots \rightarrow H_{p+1}(X_b) & \xrightarrow{\delta_p^b} & H_p((U \cap V)_b) \end{array}$$

$$\begin{array}{ccc} \xrightarrow{\varphi_p^a} H_p(U_a) \oplus H_p(V_a) & \xrightarrow{\sigma_p^a} & H_p(X_a) \rightarrow \cdots \\ \downarrow \beta_p & & \downarrow \gamma_p \\ \xrightarrow{\varphi_p^b} H_p(U_b) \oplus H_p(V_b) & \xrightarrow{\sigma_p^b} & H_p(X_b) \rightarrow \cdots \end{array}$$

where the horizontal lines belong to the usual Mayer-Vietoris sequence of the triads (X_a, U_a, V_a) and (X_b, U_b, V_b). The horizontal homomorphisms are defined as $\delta_p^a([x]) = [\partial(x|_U)], \varphi_p^a([y]) = ([y], [-y])$ and $\sigma_p^a([z], [\acute{z}]) = [z + \acute{z}]$.

Lemma 1 (Fabio & Landi, 2011)**:**
Each horizontal line in the above diagram is exact. Moreover, each square in the same diagram is commutative.

By the definition, we have for every $p \in \mathbb{Z}$ and every $a < b \in \mathbb{N}$,
$(i)\ Im\ \alpha_p = H_p^{a,b}(U \cap V)$,
$(ii)\ Im\ \beta_p = H_p^{a,b}(U) \oplus H_p^{a,b}(V)$,
$(iii)\ Im\ \gamma_p = H_p^{a,b}(X)$.
The following proposition states that the commutativity of squares in the above diagram induces a Mayer-Vietoris sequence of order 2 involving the p-th persistent homology groups for every integer p.

Proposition 2 (Fabio & Landi, 2011)**:**
Consider the sequence of homomorphisms in persistent homology groups

$$\cdots \to H_{p+1}^{a,b}(X) \overset{\delta}{\to} H_p^{a,b}(U \cap V) \overset{\varphi}{\to} H_p^{a,b}(U) \oplus H_p^{a,b}(V) \overset{\sigma}{\to} H_p^{a,b}(X) \to \cdots$$

where $\delta = \delta_p^b|_{Im\,\gamma_{p+1}}, \varphi = \varphi_p^b|_{Im\,\alpha_p}$ and $\sigma = \sigma_p^b|_{Im\,\beta_p}$ is of order 2. i.e.
$(i)\ Im\ \delta \subseteq Ker\ \varphi$,
$(ii)\ Im\ \varphi \subseteq Ker\ \sigma$,
$(iii)\ Im\ \sigma \subseteq Ker\ \delta$.
Next, we show that the Mayer-Vietoris sequence for graded persistence modules is exact. Before we state the theorem, let $s_p^a: H_p((U \cap V)_a) \to H_p(U_a)$ be defined by $s_p^a([y]) = [y]$ and also $t_p^a: H_p((U \cap V)_a) \to H_p(V_a)$ be defined by $t_p^a([y]) = [-y]$.
Thus $\varphi_p^a([y]) = \left(s_p^a([y]), t_p^a([y])\right)$.

Theorem 2:
The Mayer-Vietoris sequence of graded persistence modules

$$\cdots \to \mathcal{H}_{p+1}(X) \overset{\delta}{\to} \mathcal{H}_p(U \cap V) \overset{\varphi}{\to} \mathcal{H}_p(U) \oplus \mathcal{H}_p(V) \overset{\sigma}{\to} \mathcal{H}_p(X) \to \cdots$$

is exact, where $\delta(n^0, n^1, \cdots, n^m) = \left(\delta_p^0(n^0), \delta_p^1(n^1), \cdots, \delta_p^m(n^m)\right)$,

$$\varphi(y^0, y^1, \cdots, y^m) = \begin{pmatrix} \left(s_p^0(y^0), s_p^1(y^1), \cdots, s_p^m(y^m)\right), \\ \left(t_p^0(y^0), t_p^1(y^1), \cdots, t_p^m(y^m)\right) \end{pmatrix},$$

$\sigma(u, v) = \left(\sigma_p^0(u^0, v^0), \sigma_p^1(u^1, v^1), \cdots, \sigma_p^m(u^m, v^m)\right)$,
For all $n^a \in H_{p+1}^a(X), y^a \in H_p^a(U \cap V), u^a \in H_p^a(U)$ and $v^a \in H_p^a(V)$.
Equivalently we have
$(i)\ Im\ \delta = Ker\ \varphi$,
$(ii)\ Im\ \varphi = Ker\ \sigma$,
$(iii)\ Im\ \sigma = Ker\ \delta$.

Proof: We only prove (i). The two other claims can be proven similarly.
Let $(y^0, y^1, \cdots, y^m) \in Im\ \delta$. Then there exists an element $(n^0, n^1, \cdots, n^m) \in \mathcal{H}_{p+1}(X)$ such that $\delta(n^0, n^1, \cdots, n^m) = \left(\delta_p^0(n^0), \delta_p^1(n^1), \cdots, \delta_p^m(n^m)\right) = (y^0, y^1, \cdots, y^m)$.
In particular, $\delta_p^a(n^a) = y^a$ and $y^a \in Im\ \delta_p^a = Ker\ \varphi_p^a$ for all a by Lemma 1.
Then we get
$\varphi_p^a(y^a) = \left(s_p^a(y^a), t_p^a(y^a)\right) = (0,0)$ and $s_p^a(y^a) = t_p^a(y^a) = 0$.
Therefore, we have

$$\begin{aligned} &\varphi(y^0, y^1, \cdots, y^m) \\ &= \begin{pmatrix} \left(s_p^0(y^0), s_p^1(y^1), \cdots, s_p^m(y^m)\right), \\ \left(t_p^0(y^0), t_p^1(y^1), \cdots, t_p^m(y^m)\right) \end{pmatrix} \\ &= \left((0,0,\cdots,0), (0,0,\cdots,0)\right) \end{aligned}$$

and $(y^0, y^1, \cdots, y^m) \in Ker\ \varphi$. Hence, we get $Im\ \delta \subseteq Ker\ \varphi$.
Let $(c^0, c^1, \cdots, c^m) \in Ker\ \varphi$. Then

$$\begin{aligned} &\varphi(c^0, c^1, \cdots, c^m) \\ &= \begin{pmatrix} \left(s_p^0(c^0), s_p^1(c^1), \cdots, s_p^m(c^m)\right), \\ \left(t_p^0(c^0), t_p^1(c^1), \cdots, t_p^m(c^m)\right) \end{pmatrix} \end{aligned}$$

$= \big((0,0,\cdots,0),(0,0,\cdots,0)\big)$.

By equalities, we get $s_p^a(c^a) = 0$ and $t_p^a(c^a) = 0$ for all $c^a \in H_p^a(U \cap V)$.

Thus, $c^a \in Ker\ \varphi_p^a = Im\ \delta_p^a$ by lemma 1. Then there exists $n^a \in H_{p+1}^a(X)$ such that $\delta_p^a(n^a) = c^a$ for all $a \geq 0$ and we get

$(c^0, c^1, \cdots, c^m) =$

$\left(\delta_p^0(n^0), \delta_p^1(n^1), \cdots, \delta_p^m(n^m)\right) =$

$\delta(n^0, n^1, \cdots, n^m)$.

Therefore, $(c^0, c^1, \cdots, c^m) \in Im\ \delta$ which implies that $Ker\ \varphi \subseteq Im\ \delta$.

Hence, $Im\ \delta = Ker\ \varphi$. The proof is complete.

5. LONG (EXACT) SEQUENCE FOR PERSISTENT HOMOLOGY

Definition 16:

Let X be a compact triangulated space, and $U \subset X$. For any $a < b \in \mathbb{N}$, we can consider the following diagram.

$$\begin{array}{ccccccc}
\cdots \to H_{p+1}(X_a, U_a) & \xrightarrow{\delta_p^a} & H_p(U_a) & \xrightarrow{\alpha_p^a} & H_p(X_a) \\
\downarrow \gamma_{p+1} & & \downarrow \varphi_p & & \downarrow \omega_p \\
\cdots \to H_{p+1}(X_b, U_b) & \xrightarrow{\delta_p^b} & H_p(U_b) & \xrightarrow{\alpha_p^b} & H_p(X_b)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\beta_p^a} H_p(X_a, U_a) & \xrightarrow{\delta_p^a} & H_{p-1}(A_a) \to \cdots \\
\downarrow \gamma_p & & \downarrow \varphi_{p-1} \\
\xrightarrow{\beta_p^b} H_p(X_b, U_b) & \xrightarrow{\delta_p^b} & H_{p-1}(U_b) \to \cdots
\end{array}$$

Where $\delta_p^a([c]) = \left[(\partial c)|_{U_a}\right]$ and α_p^a, β_p^a are homomorphisms induced by inclusion and quotient maps, respectively.

Lemma 2: Each horizontal line in the above diagram is exact. Moreover, each square in the same diagram is commutative.

Proof: This follows from the exactness of the relative homology sequence of the pairs (X_a, U_a) and (X_b, U_b) and naturally from the diagram.

Note that for every $a < b \in \mathbb{N}$, $(X_a, U_a) \subset (X_b, U_b)$ gives a filtration of (X, U), since $X_a \subset X_b$ and $U_a \subset U_b$. We denote $(X, U)_a := (X_a, U_a)$ and $H_p^a((X, U)) := H_p((X, U)_a) := H_p(X_a, U_a)$. By definition of persistent homology groups, we get

$(i)\ Im\ \varphi_p = H_p^{a,b}(U)$

$(ii)\ Im\ \omega_p = H_p^{a,b}(X)$

$(iii)\ Im\ \gamma_p = H_p^{a,b}(X, U)$.

Proposition 3:

The sequence of homomorphisms of persistent homology groups

$$\cdots \to H_{p+1}^{a,b}(X, U) \xrightarrow{\delta} H_p^{a,b}(U) \xrightarrow{\alpha} H_p^{a,b}(X) \xrightarrow{\beta} H_p^{a,b}(X, U) \to \cdots$$

where $\delta = \delta_p^b|_{Im\ \gamma_{p+1}}$, $\alpha = \alpha_p^b|_{Im\ \varphi_p}$, $\beta = \beta_p^b|_{Im\ \omega_p}$, is exact of order 2. That is,

$(i)\ Im\ \delta \subseteq Ker\ \alpha$,

$(ii)\ Im\ \alpha \subseteq Ker\ \beta$,

$(iii)\ Im\ \beta \subseteq Ker\ \delta$.

Proof: By lemma 1, we have $Im\ \delta \subset Im\ \varphi_p, Im\ \alpha \subset Im\ \omega_p, Im\ \beta \subset Im\ \gamma_p$.

We only prove (i). The claims (ii) and (iii) can be proven analogously.

Let $c \in Im\ \delta$. Then there exists $d \in H_{p+1}(X_a, U_a)$ such that $\delta(d) = \delta_p^b\left(\gamma_{p+1}(d)\right) = c$.

Thus, we get $c \in Im\ \delta_p^b = Ker\ \alpha_p^b$. Since $c \in Im\ \varphi_p$, there exists $e \in H_p(u_a)$ such that

$\varphi_p(e) = c$. Then

$0 = \alpha_p^b(c) = \alpha_p^b\left(\varphi_p(e)\right) = \alpha_p^b|_{Im\ \varphi_p}(c) = \alpha(c)$.

Therefore, $c \in Ker\ \alpha$. Hence, $Im\ \delta \subseteq Ker\ \alpha$.

The above proposition shows that the sequence of homomorphisms in the persistence homology group is exact order two. Now, we prove that the long exact sequence for graded persistence modules of a pair (X, U) is exact. Let
$\mathcal{H}_p(X,U) = \oplus_{a=0}^{m} H_p^a(X,U)$, $\mathcal{H}_p(X) = \oplus_{a=0}^{m} H_p^a(X)$ and $\mathcal{H}_p(U) = \oplus_{a=0}^{m} H_p^a(U)$
be the graded persistence modules of $(X,U), X$, and U, respectively.

Theorem 3 (Varli et al., 2018)**:**
The long sequence of graded persistence modules of pair (X, U)

$$\cdots \to \mathcal{H}_{p+1}(X,U) \overset{\delta}{\to} \mathcal{H}_p(U) \overset{\alpha}{\to} \mathcal{H}_p(X) \overset{\beta}{\to} \mathcal{H}_p(X,U) \to \cdots$$

is exact, where $\delta(n^0, n^1, \cdots, n^m) = \left(\delta_p^0(n^0), \delta_p^1(n^1), \cdots, \delta_p^m(n^m)\right)$,
$\alpha(y^0, y^1, \cdots, y^m) = \left(\alpha_p^0(y^0), \alpha_p^1(y^1), \cdots, \alpha_p^m(y^m)\right)$,
$\beta(x^0, x^1, \cdots, x^m) = \left(\beta_p^0(x^0), \beta_p^1(x^1), \cdots, \beta_p^m(x^m)\right)$.
That is,
(i) $Im\ \delta = Ker\ \alpha$,
(ii) $Im\ \alpha = Ker\ \beta$,
(iii) $Im\ \beta = Ker\ \delta$.

Proof: We prove only (i). The other two claims can be obtained analogously.
Let $(y^0, y^1, \cdots, y^m) \in Im\ \delta$. Then there exists an element $(n^0, n^1, \cdots, n^m) \in \mathcal{H}_{p+1}(X)$ such that $\delta(n^0, n^1, \cdots, n^m) = \left(\delta_p^0(n^0), \delta_p^1(n^1), \cdots, \delta_p^m(n^m)\right) = (y^0, y^1, \cdots, y^m)$.
So $\delta_p^a(n^a) = y^a$ and $y^a \in Im\ \delta_p^a = Ker\ \alpha_p^a$ for all a by lemma 2.
Then we get $\alpha_p^a(y^a) = 0$.
Therefore, $\alpha(y^0, y^1, \cdots, y^m) = \left(\alpha_p^0(y^0), \alpha_p^1(y^1), \cdots, \alpha_p^m(y^m)\right) = (0, 0, \cdots, 0)$
and $(y^0, y^1, \cdots, y^m) \in Ker\ \alpha$. Thus, $Im\ \delta \subset Ker\ \alpha$.
Let $(c^0, c^1, \cdots, c^m) \in Ker\ \alpha$. Then
$\alpha(c^0, c^1, \cdots, c^m) = \left(\alpha_p^0(c^0), \alpha_p^1(c^1), \cdots, \alpha_p^m(c^m)\right) = (0, 0, \cdots, 0)$.
We get $\alpha_p^a(c^a) = 0$ for all $c^a \in H_p^a(U)$. Thus, $c^a \in Ker\ \alpha_p^a = Im\ \delta_p^a$ by lemma 2. Then there exists $n^a \in H_{p+1}^a(X,U)$ such that $\delta_p^a(n^a) = c^a$ for all $a \geq 0$ and we get $(c^0, c^1, \cdots, c^m) = \left(\delta_p^0(n^0), \delta_p^1(n^1), \cdots, \delta_p^m(n^m)\right) = \delta(n^0, n^1, \cdots, n^m)$.
Therefore, $(c^0, c^1, \cdots, c^m) \in Im\ \delta$ which implies that $Ker\ \alpha \subseteq Im\ \delta$.
Hence, $Im\ \delta = Ker\ \alpha$. The proof is complete.

6. STABILITY OF PERSISTENT HOMOLOGY

In this section, we explain the stability theorem for persistent modules. Let $\mathfrak{E} = \{K_\alpha\}_{\alpha \in R}$ and $\mathfrak{F} = \{L_\alpha\}_{\alpha \in R}$ be two filtrations of simplicial complexes graded over $\mathbb{R}$. Let $\mathcal{A}$ and $\mathcal{B}$ be their persistence modules over PID R, respectively. So that $\mathcal{A} = \{H(K_\alpha)\}_{\alpha \in R}$ and $\mathcal{B} = \{H(L_\alpha)\}_{\alpha \in R}$, where each family is joined by homomorphisms $a_\alpha^\beta = H(K_\alpha) \to H(K_\beta)$ and $b_\alpha^\beta = H(L_\alpha) \to H(L_\beta)$; $(\alpha < \beta)$, respectively, which are induced by the inclusion maps. Since we took persistence modules over a PID R, every persistence module can be decomposed by Theorem 1.

Definition 17 (Davies, 2019):
Two persistent modules $\mathcal{A}$ and $\mathcal{B}$ are said to be $\in -$*interleaved* if Figure 9 is commutative for all $u \in \mathbb{R}$.

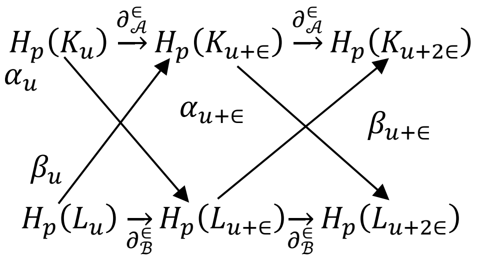


Figure 9. Commutative Diagram of the Two Persistent Modules $\mathcal{A}$ and $\mathcal{B}$.

Where K and L are simplicial complexes.

Definition 18:
The *interleaving distance* between two persistence modules $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$d_i(\mathcal{A}, \mathcal{B}) = inf\{\in | \mathcal{A}, \mathcal{B} \text{ are } \in -interleaved\}$$

or $d_i(\mathcal{A}, \mathcal{B}) = \infty$ if there is no $\in -$interleaving between $\mathcal{A}$ and $\mathcal{B}$.

Interpolation lemma (Chazal et al., 2016) states that suppose $\mathcal{A}, \mathcal{B}$ are a $\delta -$interleaved pair of persistence modules. Then there exists one parameter family of persistence modules $(\mathcal{A}_x | x \in [0, \delta])$ such that $\mathcal{A}_0, \mathcal{A}_\delta$ are equal to $\mathcal{A}, \mathcal{B}$, respectively, and $\mathcal{A}_x, \mathcal{A}_y$ are $|y - x| -$interleaved for all $x, y \in [0, \delta]$.

Theorem 4 (Davies, 2019):
If $\mathcal{A}$ and $\mathcal{B}$ are persistent modules, then

$$d_b\big(Dgm(\mathcal{A}), Dgm(\mathcal{B})\big) \leq d_i(\mathcal{A}, \mathcal{B}).$$

Proof: This is equivalent to showing that whenever $\mathcal{A}$ and $\mathcal{B}$ are persistent modules that are $\in -$interleaved, there exists an $\in -$matching between $Dgm(\mathcal{A})$ and $Dgm(\mathcal{B})$. This statement is a special case of a stability theorem for a specific type of measure. In particular, an r-measure is a measure μ that maps the set of all rectangles $[s, t] \times [u, v]$ in a subset plane to $\mathbb{N} \cup \{\infty\}$.
The r-measure of a persistence module $\mathcal{B}$ is defined on a rectangle $R = [s, t] \times [u, v]$ by

$$\mu_{\mathcal{B}}(R) = \begin{cases} 1 & [t,u] \subseteq (i,j) \subseteq (s,v) \\ 0 & otherwise \end{cases}$$

where $Q(i, j)$ is a summand in the decomposition of $\mathcal{B}$.
The interpolation lemma states that if $\mathcal{A}$ and $\mathcal{B}$ are $\in -$interleaved persistent modules, then there exists a one-parameter family of persistence modules $\{\mathcal{A}_x | x \in [0, \delta]\}$ with $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_\delta = \mathcal{B}$ such that $\mathcal{A}_x$ and $\mathcal{A}_y$ are $\in -$interleaved for all $x, y \in [0, \delta]$.
If $R = [s, t] \times [u, v]$ is a rectangle in the plane, then the $\in -$thickening of R is $R^{\in} = [s - \in, t + \in] \times [u - \in, v + \in]$. Then the stability theorem for finite r-measure (see in Chazal et al. (2016)) states that if $\{\mu_x | x \in [0, \in]\}$ is a one-parameter family of finite r-measures on an open set $\mathcal{D} \subseteq \mathbb{R}^2$, then if for all $x, y \in [0, \delta]$ we have $\mu_x(R) \leq \mu_y\big(R^{|y-x|}\big)$ holds for every rectangle whose $|y - x| -$thickening is contained in $\mathcal{D}$, then there exists an $\in -$matching between the diagrams representing μ_x and μ_y contained in $\mathcal{D}$.
Now, suppose $\mathcal{A}$ and $\mathcal{B}$ are $\in -$interleaved persistent modules. Then the Interpolation lemma implies that there exists a one parameter family of persistence modules such that $\mathcal{A}_x$ and $\mathcal{A}_y$ are $\in -$interleaved for all $x, y \in [0, \delta]$. Then we link this one parameter family to the one-parameter family of r-measures. So, this proves the necessary conditions. After proving these necessary conditions, the Stability Theorem for finite r-measures implies that there exists an $\in$

−matching between the diagrams $\mathcal{A}$ and $\mathcal{B}$. Taking the infimum of $\in$ gives the Stability Theorem.

7. DIAMOND PRINCIPLE OF PERSISTENT HOMOLOGY

Definition 19:

Let $\mathbb{B}$ be a sequence of vectors and linear maps of length n:

$$B_1 \overset{K_1}{\leftrightarrow} B_2 \overset{K_2}{\leftrightarrow} \cdots \overset{K_{n-1}}{\longleftrightarrow} B_n$$

Each $\overset{K_i}{\leftrightarrow}$ represents either a forward map $\overset{f_i}{\rightarrow}$ or a backward map $\overset{g_i}{\leftarrow}$. Then the object $\mathbb{B}$ is called a *zigzag diagram* of vector spaces, or simply a *zigzag module* over a field $\mathbb{R}$. (Zomorodian & Carlsson, 2005) Persistence modules are zigzag modules when all the maps have a forward orientation.

Definition 20:

Consider σ is a type of length n and integers $1 \leq b \leq d \leq n$. Then the interval σ −module with birth time b and death time d is written as $\mathbb{I}_\sigma(b, d)$ and defined with spaces,

$$I_j = \begin{cases} \mathbb{R} & if\ b \leq j \leq d \\ 0 & otherwise \end{cases}$$

and with identity maps between adjacent copies of $\mathbb{R}$ and zero maps otherwise. When σ is implicit, we simply write $\mathbb{I}(b, d)$. Let $\mathbb{B}$ be a zigzag module of arbitrary type. Then the *zigzag persistence* of $\mathbb{B}$ is defined to be the multi-set

$\mathrm{Pers}(\mathbb{B}^+) =$
$\{(b_i, d_i) \subseteq \{1,2,\cdots,n\} | i = 1,2,\cdots,N\}$
of integer intervals derived from a decomposition
$\mathbb{B} \cong \mathbb{I}(b_1, d_1) \oplus \cdots \oplus \mathbb{I}(b_N, d_N)$.

Definition 21:

Consider Figure 10.

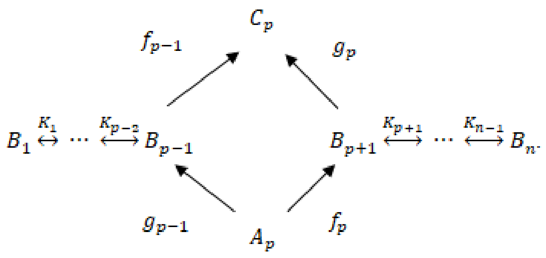


Figure 10. A diagram on $\mathbb{B}$

Let $\mathbb{B}^+$ and $\mathbb{B}^-$ denote the two zigzags modules contained in Figure 11.

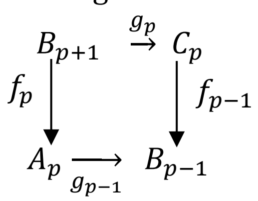
$$\mathbb{B}^+ = (B_1 \overset{K_1}{\leftrightarrow} \cdots \overset{K_{p-2}}{\longleftrightarrow} B_{p-1} \overset{f_{p-1}}{\longrightarrow} C_p \overset{g_p}{\leftarrow} B_{p+1} \overset{K_{p+1}}{\longleftrightarrow} \cdots \overset{K_{n-1}}{\longleftrightarrow} B_n)$$

$$\mathbb{B}^- = (B_1 \overset{K_1}{\leftrightarrow} \cdots \overset{K_{p-2}}{\longleftrightarrow} B_{p-1} \overset{g_{p-1}}{\longleftarrow} A_p \overset{f_p}{\rightarrow} B_{p+1} \overset{K_{p+1}}{\longleftrightarrow} \cdots \overset{K_{n-1}}{\longleftrightarrow} B_n)$$

Figure 11. The two zigzags modules $\mathbb{B}^+$ and $\mathbb{B}^-$

If we compare $\mathrm{Pers}(\mathbb{B}^+)$ with $\mathrm{Pers}(\mathbb{B}^-)$, particularly with respect to intervals that meet $\{p-1, p,\ p+1\}$, this requires a favorable condition on the four maps in the *middle diamond*.

We say that the diagram



is exact, if $Im(D_1) = Ker(D_2)$ in the following sequence

$$A_p \overset{D_1}{\rightarrow} B_{p-1} \oplus B_{p+1} \overset{D_2}{\rightarrow} C_p$$

where $D_1(a) = g_{p-1}(a) \oplus f_p(a)$ and $D_2(b \oplus \acute{b}) = f_{p-1}(b) - g_p(\acute{b})$.

CONCLUSIONS AND SUGGESTIONS

An extensive review of persistent homology has been undertaken, encompassing a comprehensive exploration of various constructions of persistent modules rooted in functors and graded modules. The theoretical scrutiny, employing Eilenberg-Steenrod axioms, has unequivocally established persistent homology as a bona fide homology theory. Essential properties such as the Mayer-Vietoris formula, long exact sequences, and the Diamond principle have been succinctly summarized, bolstering our understanding of the intricacies of topological data analysis. The stability of persistent homology has also given rise to cutting-edge innovations such as persistence landscapes, which offer both robustness and seamless integration with prevalent machine learning and statistical methodologies (Bubenik, 2015).

In the current era, characterized by the profound influence of the fourth industrial revolution, nearly every industry—ranging from agriculture to government sectors—has become a prolific generator of vast digital datasets. In these domains, the adept analysis of intricate geometric data assumes paramount significance, directly impacting areas such as medical research, drug discovery, cybersecurity, and beyond. By harnessing the sophisticated techniques of persistent homology, society can benefit immensely from its applications, solving practical challenges and unlocking substantial economic value. This convergence of mathematical theory and practical application underscores the potential for persistent homology to play a pivotal role in shaping the future of data-driven decision-making across a multitude of industries. In particular, this foundational study serves as a launchpad for unearthing new properties of persistent homology, paving the way for the creation of efficient tools in the realm of topological data analysis. This review effort encourages more investigation into persistent homology, particularly with regard to its properties. More studies will open up new foundations for persistent homology, which is the main objective of this field of study.

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