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Expansion $\overset{\frown}{gis}$-closed & its lower separation axioms

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ABSTRACT

In this paper, we introduced a new classification of generalized closed sets that are called ultra-generalized s-closed sets named simply $\overset{\frown}{gis}$-closed sets, and we show the relationship between this new type with other open, and generalized closed sets. We also investigate advanced $\overset{\frown}{gis}$ - continuous mappings and some of its properties. Furthermore, we discussed some lower $\overset{\frown}{gis}$-seperation axioms.

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INTRODUCTION

Contains Introduction many open, closed generalized closed sets will not stop at certain limit, because we live in infinite evaluation. So many researchers gave us their vision in creation or existence their open and closed sets; they also introduced the applications to those sets. By the way our simple working in this paper aim to create new types of closed set sets that are called $\overset{\frown}{gis}$-closed set and we present expansion of $\overset{\frown}{gis}$-continuous mapping. Furthermore, we show the investigation properties of these mappings. In section 2, we define $\overset{\frown}{gis}$-open set, and we appear the relationship with some types of generalized closed and open sets. In section 3, we present the notion of $\overset{\frown}{gis}$-continuous mapping, $\overset{\frown}{gis}$-open mapping, $\overset{\frown}{gis}$-irresolute mapping, and $\overset{\frown}{gis}$-homeomorphism mapping, and we investigate the relationship between the new type of generalized mapping with some types of continuous mappings, the relationship between $\overline{is}$-open mapping, with some types of open mappings and the relationship between $\overline{is}$-irresolute mappings with some types of irresolute mappings. Further, we compare $\widetilde{\overline{gis}}$-homeomorphism with other kind of homeomorphisms. Further, we study some of their basic properties.

METHOD

Our needed in this paper to suggest renaming the topology spaces to (X, τ), and (Y, σ) by τ_o and σ_o respectively. We denoted the following notations, characterizations, and definitions. The closure (resp. interior) of a subset B of a topological space τ_o is called by $CL(B)$(resp.$Int(B)$), and the complementation of B is represented by $\bar{B}$.

Definition 1.1 A subset B of a topological space τ_o is said to be (i) Semi-open set (Levine, 1959) (resp. α–open set (Njastad, 1965), regular-open set (Levine, 1970), iα-open set (Mohammed & kahtab, 2012), and $\overline{is}$-open set (Y. khattab, 2022)) , if $B \subseteq Cl(Int(B))$ (resp. $B \subseteq Int(Cl(B))$, $B = int(cl(B))$, $B \subseteq Cl(B \cap O)$, where $\exists O \in \alpha O(X)$,$O \neq X, \emptyset$, and $B \subseteq Cl(B \cap O)$, where $\exists O \neq \emptyset, O \subseteq SO(X)$.

The family of all open (resp. semi-open,α-open, regular open, iα-open, and $\overline{is}$-open) sets of a topological space is denoted by $\widetilde{\tau}$, (resp. τ_s, τ_α, τ_r, $\tau_{i\alpha}$, and $\tau_{\overline{is}}$). The complementation of open (resp. α-open, semi-open, regular-open, iα-open, and $\overline{is}$-open) sets of a topological space X is called closed (resp. α-closed, semi-closed, regular-closed, iα closed, $\overline{is}$-closed) sets. The closure of the above sets is denoted by Cl (resp. Cl_α, $Cl_{s,}$, Cl_r, and $Cl_{i\alpha}$,and $Cl_{\overline{is}}$).

Definition 1.2 Let $B \subseteq \tau_o$, B is said to be

(i) Generalized closed set (simply, g-closed (Maki et al., 1993)), if $cl(B) \subseteq O$, whenever $B \subseteq O$ and $O \subset \tau_o$.
(ii) Semi-generalized closed (simply, sg-closed) (Bhattacharya & Lahiri, 1987), if $cl(B) \subseteq O$, whenever $B \subseteq O$and $O \subset \tau_s$.
(iii) *Generalized semi-closed (*simply*, gs-closed)* (Arya & Nour, 1990)*, if $Cl_{s,}(B) \subseteq O$ whenever $B \subseteq O$ and $O \subset \tau$;*
(iv) *Generalized α-closed (*simply*, gα-closed)* (Maki et al., 1993)*, if $Cl_\alpha(B) \subseteq O$ whenever $B \subseteq O$ and $O \subset \tau_\alpha$;*
(v) *α-generalized closed (*simply*, αg-closed)* (Maki et al., 1994)*, if $Cl_\alpha(B) \subseteq O$ whenever $B \subseteq O$ and $O \subset \tau$;*
(i) *Regular generalized closed (*simply*, r-g-closed)* (Njastad, 1965)*, if $Cl(B) \subseteq O$ whenever $B \subseteq O$ and $O \subset \tau_r$.*
(ii) Generalized a regular-closed set (simply, gαr-closed) (Sekar & Kumar, 2016), if $Cl_\alpha(B) \subseteq O$ whenever $B \subseteq O$ and $O \subset \tau_s$.

Definition 1.3 The mapping $f: \tau_o \to \sigma_0$ we called:

(i) g-continuous (resp. sg-continuous, gα-continuous, iα-continuous, r-g-continuous, and $\overline{is}$-continuous) if the inverse image of every open subset of σ_o is g-open (resp. semi g-open, gα-open, iα-open, r-g-open, and $\overline{is}$-open)set in τ_o (Cao et al., 2002; Darwesh & Hassan, 2015; Devi et al., 1997; Mohammed & kahtab, 2012; Rani & Balachandran, 1997; Y. khattab, 2022).
(ii) α-irresolute (resp. r-g-irresolute, sg-irresolute, and $\overline{is}$-irresolute)if the inverse image of every α-open (resp. r-g-open, sg-open , and $\overline{is}$-open) subset of σ_o is an α-open(resp. r-g- open , sg-open , and $\overline{is}$-open) subset in τ_o (Maheswari & Prasad, 1978; Rani & Balachandran, 1997; Sundaram et al., 1991; Y. khattab, 2022).
(iii) regular open (resp. rg-open, iα-open, and $\overline{is}$-open) if image of each open set of τ_o is regular –open(resp. rg-open, iα-open, and $\overline{is}$-open) in σ_o (Mohammed & kahtab, 2012; Sekar & Kumar, 2016; Stone, 1937; Y. khattab, 2022).

Definition 1.4 The mapping $f: \tau_o \to \sigma_0$ is said to be :

(i) g-homeomorphism (resp. gc-homeomorphism, $i\alpha$- homeomorphism, and $\overline{is}$- homeomorphism) if f is g-continuous and g-open (resp. f and $f^{-1}$are g-irresolute, if f is an $i\alpha$-continuous and open, and f is an $\overline{is}$-continuous and open mappings) (Maki et al., 1991, 1994; Mohammed & kahtab, 2012; Y. khattab, 2022).

Definition 1.5 Let (X, τ_o) topology space, (X, τ_o) then is defined to be:

(i) $\overline{is}$-T_0 (Y. khattab, 2022) if $\forall\, n \neq m \in X$, $\exists\, B \subseteq \tau_{\overline{is}}$, s.t $n \in B \wedge m \notin B$.

(ii) $\overline{is}$-T_1 (Y. khattab, 2022) if $\forall\, n \neq m \in X$, $\exists\, B \wedge C \subseteq \tau_{\overline{is}}$, s.t $n \in B \wedge m \notin B, m \in C \wedge n \notin C$.

(iii) $\overline{is}$-T_2 (Y. khattab, 2022) $\forall\, n \neq m \in X$, $\exists\, B \neq C \subseteq \tau_{\overline{is}}$, s.t $n \in B, \wedge m \in C$.

(iv) $\overline{is}$–regular [10] if $\forall\, n \notin \lambda \subseteq \overline{\tau_o}, \exists\, B \neq C \subseteq \tau_{\overline{is}}$, s.t $n \in B \wedge \lambda \subseteq C$.

(v) $\overline{is}$–normal (resp. ultra-normal, s-normal, α-normal) $\bigvee B \neq C \subseteq \overline{\tau}_0, B \subseteq M \subseteq \tau_{\overline{is}} (resp. \tau_s, \quad \tau_\alpha, \quad \tau_r) \neq C \subseteq N \subseteq \tau_{\overline{is}}\ (resp. \tau_s, \quad \tau_\alpha, \quad \tau_r)$ (Arhangel'skii & Ludwig, 2001; Maheswari & Prasad, 1978; Staum, 1974; Y. khattab, 2022).

(vi) $\overline{is}$-$T_{1/2}$ (Y. khattab, 2022) if $\overline{\tau_{\overline{is}}} = \overline{\tau}_{\overline{i}}$.

RESULTS AND DISCUSSION

$\widetilde{gis}$-Closed (Open) Sets & Its Applications

Definition 2.1 Let $B \subseteq \tau_o$, B is said to be ultra- generalized S-closed set simply ($\widetilde{gis}$- closed) , if $Cl_{\overline{is}}(B) \subseteq O$, whenever $B \subseteq O, \wedge O \subset \tau_{\overline{is}}$. The complementation of $\widetilde{gis}$- closed set is $\widetilde{gis}$- open set, then collections of $\widetilde{gis}$-open sets are denoted by $\tau_{\widetilde{gis}}$.

Example 2.2 X={0,4,8}, τ={∅, {8},X } ,$\tau_{\widetilde{gis}}$={ ∅,{0}, {4}, {8}, {0,4}, {0,8}, {4,8}, ,X}.

Lemma 2.3 If $B \subseteq \tau(resp.\ \tau_{s,} \tau_\alpha, and\ \tau_{i\alpha})$, then B is $\tau_{\overline{is}}$ (Y. khattab, 2022).

Lemma 2.4 Every $\overline{is}$-closed set is $\widetilde{gis}$-closed sets.

Proof. Let $B \subseteq \tau_{\overline{is}}$ such that $B \subseteq U$, where U is $\overline{is}$- open. Since $Cl_{\overline{is}}(B) \subseteq B$. It implies $Cl_{\overline{is}}(B) \subseteq U$. Therefore, B is $\widetilde{gis}$-closed set in τ_o ■

Theorem 2.5 Every closed set is $\widetilde{gis}$ -closed.

Proof. Let $V^c$is open set in a topological spaceτ_o. Since every open set is $\overline{is}$-open set by **Lemma 2.3** , then V^c is $\overline{is}$-open set, consequently V is $\overline{is}$-closed set, since every $\overline{is}$-closed set is $\widetilde{gis}$-closed sets by **Lemma 2.4**, Therefore V is $\widetilde{gis}$-closed set■

Corollary 2.6 Every semi-closed is $\widetilde{gis}$-open**.**

Proof. Let N^c is semi-open set in τ_o. Since every semi–open set is $\overline{is}$-open by **Lemma 2.3**, hence N is $\widetilde{gis}$-closed by **Lemma 2.4**. Therefore, N is $\widetilde{gis}$-open■

Corollary 2.7 Every α-closed is $\widetilde{gis}$-open**.**

Proof. Same the proof **Corollary 2.6** by **Lemma 2.3**, and **Lemma 2.4** ■

Corollary 2.8 Every $i\alpha$-closed is $\widetilde{gis}$-open**.**

Proof. Same the proof **Corollary 2.6** by **Lemma 2.3**, and **Lemma 2.4** ■

Generally, the converse of above theorems and corollaries are not true as this example.

Example 2.9 X={0,2,4}, τ=τ_α={∅,{2,4},X}, $\tau_{\overline{is}}$={ ∅, {4}, {2}, {2,4}, {0,2}, {0,4},X}, and $\overline{is}$- closed sets ={ ∅,{2,0}, {0,4}, {0}, {4}, {2},X} we get {2,4}⊆ {2,4}, $Cl_{\overline{is}}$\{2,4\} ⊆ {2,4}, Therefore {a} is $\widetilde{gis}$-closed but it is not $\overline{is}$-closed set, semi-closed, α-closed, and closed .

Theorem 2.10 Every g-closed is $\widetilde{gis}$-closed.

Proof. Let V is sg-closed in a topological τ_o, such that $Cl(V) \subseteq O$,Such that V ▯ $O \subseteq \tau_o$, Since every closed is an $\overline{is}$-closed by **Lemma 2.3**, then $Cl_{\overline{is}}(V) \subseteq$

$Cl(V) \subseteq O$, hence $Cl(V) \subseteq O$. Also, since every open set is an $\overline{is}$-open **Lemma 2.3,** then $V \subseteq O \subseteq \tau_{\overline{is}}$. Therefore, V is $\widehat{gis}$-closed■

Theorem 2.11 Every sg-closed set is $\widehat{gis}$-closed set.

Proof. Let V is sg-closed in τ_o, consequently $Cl(V) \subseteq O$, Such that $V \subseteq O \subseteq \tau_s$, Since every closed is an $\overline{is}$-closed by **Lemma 2.3,** then we have $Cl(V) \subseteq Cl_{\overline{is}}(V) \subseteq O$. Also, since every semi-open is an $\overline{is}$-open**, Lemma 2.3,** then $V \subseteq O \subseteq \tau_{i\alpha}$. Therefore, V *is* $\widehat{gis}$-closed■

Corollary 2.12 Every gs-closed is $\widehat{gis}$-closed.

Proof. Same the proof of above theorem ■

Theorem 2.13 Every $g\alpha$-closed is $\widehat{gis}$-closed.

Proof. Let V is αg- closed in τ, then $Cl(V)_{\alpha} \subseteq O$, Such that $V \subseteq O \subseteq \tau_{\alpha}$, Since every α-closed is $\overline{is}$-closed **Lemma 2.3,** then $Cl(V)_{\alpha} \subseteq Cl_{\overline{is}} \subseteq O$, and since every α-open is an $\overline{is}$-open by **Lemma 2.3,** then $V \subseteq O \subseteq \tau_{\overline{is}}$. Therefore, V *is* $\widehat{gis}$-closed■

Corollary 2.14 αg- closed is $\widehat{gis}$-closed.

Proof. Same the proof of the above theorem■

Remark 2.15 the following example show that $\widehat{gis}$-closed set isn't an essential to be g-closed, gs-closed, set sg-closed, αg-closed, and gα-closed set.

Example 2.16 X={i,j,k}, τ={∅,{j},X}$\tau_{\alpha} = \tau_s = \{\emptyset, \{j\}, \{j,i\}\{j,k\}, X\}$, $\tau_{\overline{is}}$={ $\emptyset, \{i\}, \{j\}, \{k\}, \{i,j\}, \{i,k\}, \{j,k\}, X\}$= $\overline{is}$- closed sets. We get {j,k}⊆ $\{j,k\}$, $Cl_{\overline{is}}\{j,k\} \subseteq \{j,k\}$, Therefore {j,k} is $\widehat{gis}$-closed but is not g-closed, gs-closed, set sg-closed, αg- closed, and gα-closed set.

Theorem 2.17 Every rg-closed is $\widehat{gis}$-cloed set.

Proof. Let V is rg-closed in τ, so we have $Cl(V) \subseteq O$, Such that $V \subseteq O \subseteq \tau_r$, Since every closed is an $\overline{is}$-closed by **Lemma 2.3,** then $Cl(V) \subseteq Cl_{\overline{is}} \subseteq O$, and since every regular-open is an $\overline{is}$-open **Lemma 2.3,** then $V \subseteq O \subseteq \tau_{\overline{is}}$. Therefore, V *is* $\widehat{gis}$-closed■

Corollary 2.18 Every αrg-closed *is* $\widehat{gis}$-cloed set.

Proof. Let V is a regular-closed -closed, so we have $Cl_{\alpha}(V) \subseteq O$, Such that $V \subseteq O \subseteq \tau_r$, Since Every α closed is $\overline{is}$-closed **Lemma 2.3,** then $Cl_{\alpha}(V) \subseteq Cl_{\overline{is}} \subseteq O$, and since every regular -open is an $\overline{is}$-open **Lemma 2.3,** then $V \subseteq O \subseteq \tau_{\overline{is}}$. Therefore, V *is* $\widehat{gis}$-closed■

Remark 2.19 the following example show that $\widehat{gis}$-closed is not a necessary to be αgr-closed, rg-closed, and set αrg-closed as the following example.

Example 2.20 X={0,5,10}, τ ={∅,{10}, X}, $\tau_s = \{\emptyset, \{10\}, \{10,0\}, \{10,5\}, X\}$, $\tau_{\overline{is}}$={ $\emptyset, \{0\}, \{5\}, \{10\}, \{10,5\}, \{10,0\}, \{0,5\}, X\}$ = $\overline{is}$- closed sets. We get {10,0}⊆ {10,0}, $Cl_{i\alpha}\{10,0\} \subseteq \{10,0\}$, Therefore {10,10} is $\widehat{gis}$-closed but is not αgr-closed, rg-closed, and set αrg-closed.

Theorem 2.21 If $\tau \subset \tau_s$, then $\tau_{\widehat{gis}} = \tau_{\overline{is}}$

Proof. If $\tau \subset \tau_s$, it implies to $\exists\ N \subset \tau_s$ *, then* $N \notin \tau$, and hence $\tau_{\overline{is}}$=po(x), equivalence to all subsets of $\overline{is}$-closed set, since every $\overline{is}$-closed set is $gi\alpha$-closed sets by **Corollary 2.4**. Therefore $\tau_{\overline{is}} = \tau_{\widehat{gis}}$■

Theorem 2.22 If M and N are $\widehat{gis}$-closed sets in (X,τ), then $M \cup N$ is $\widehat{gis}$-closed set in
(X,τ).

Proof. Let M and N are $\widehat{gis}$-closed sets in τ and U be an $\overline{is}$-open set containing M and N. Therefore $Cl(M) \subseteq U, Cl\,(N) \subseteq U$. Since M ⊆ U, N ⊆ U then $M \cup N \subseteq U$. Hence $Cl\,(M \cup N) = Cl\,(M) \cup Cl\,(N) \subseteq U$. Therefore $M \cup N$ is $\widehat{gis}$-closed set in τ■

Theorem 2.23 If M and N are $\widehat{gis}$-closed sets in(X,τ), then $M \cap N$ is $\widehat{gis}$-closed set in
(X,τ).

Proof. Same the proof of above theorem.

Some $\widehat{gis}$-Mappings & its properties

Definition 3.1 The mapping $f: \tau_o \rightarrow \sigma_0$ is $\widehat{gis}$-continuous if, $\forall\, V \subseteq \sigma_0, f^{-1}(V) \subseteq \tau_{o\widehat{gis}}$, .

Example 3.2 Let N=L= {0.1, 0.2, 0.3}, $\tau = \{\emptyset, \{0.2\}, N\}$, $\sigma = \{\emptyset, \{0.1\}, L\}$, $\widehat{gis}\,(N) = \{\emptyset, \{a\}, \{d\}, \{h\}, \{a,d\}, \{a,h\}, \{d,h\}, N\}$. It is obvious; the identity mapping $f: (N,\tau) \rightarrow (L,\sigma)$ is $\widehat{gis}$-continuous.

Theorem 3.3 If f is sg-continuous mapping, then f is $\widehat{gis}$-continuous mapping.

Prof. Let $f: \tau_o \rightarrow \sigma_0$ be sg-continuous mapping, and $N \subseteq \sigma_0(\overline{\sigma_0})$. Since, f is sg-continuous, then $f^{-1}(V)$ is an sg-open (closed) $\subseteq \tau_o$. Since, every sg-open set is an $\widehat{\text{gis}}$- -open set by **Theorem 2.11**, then $f^{-1}(N) \subseteq \tau_{o\widehat{\text{gis}}}$. Therefore, f is an $\widehat{\text{gis}}$-continuous■

Theorem 3.4 If f is gα-continuous mapping, then f is $\widehat{\text{gis}}$-continuous mapping.

Proof. Same the proof above By **Corollary 2.13**■

Theorem 3.5 Every r-g-continuous mapping is $\widehat{\text{gis}}$-continuous mapping.

Proof. Same the proof of **Theorem 3.3** and by using b**Theorem 2.17**. Therefore, f is an $\widehat{\text{gis}}$-continuous■

Theorem 3.6 Every iα-continuous mapping is $\widehat{\text{gis}}$-continuous mapping.

Proof. Same the proof of the above theorem , since every $\overline{\text{is}}$-open set is $\widehat{\text{gis}}$-open by **Corollary 2.8**■

Theorem 3.7 Every $\overline{\text{is}}$-continuous mapping is $\widehat{\text{gis}}$-continuous mapping.

Proof. Same the prooof of the above **Theorem 3.5** by using **Lemma 2.4**■

$f^{-1}(V)$ is an sg-open (closed) set in τ_o. Since, every sg-open set is an $\widehat{\text{gis}}$- -open set by **Theorem 2.11** , then $f^{-1}(V) \subseteq \tau_{o\widehat{\text{gis}}}$. ■

Theorem 3.8 If f is gα-continuous mapping, then f is $\widehat{\text{gis}}$-continuous mapping.

Proof. Same the proof above By **Corollary 2.13**■

Theorem 3.9 if f is r-g-continuous mapping, then f is $\widehat{\text{gis}}$-continuous mapping.

Proof. Let f: $\tau_o \rightarrow \sigma_0$ be an r-g-continuous mapping and $N \subseteq \sigma_0$. Since, f is r-g-continuous, then $f^{-1}(V)$ is an r-g-open set in τ, hence $f^{-1}(N) \subseteq \tau_{o\widehat{\text{gis}}}$ since every r-g-open set is an $\widehat{\text{gis}}$-open set by **Theorem 2.17**. Therefore, f is an $\widehat{\text{gis}}$-continuous■

The converse of above theorems is not ture generally.

Example 3.10 Let H={1,3,5}, K={3,6,9}, $\tau = \{\emptyset, \{1,5\}, \text{H}\}, \sigma = \{\emptyset, \{6\}, \text{K}\}$, $\widehat{\text{gis}}$-(H)={ ∅, {1}, {3}, {5}, {1,3}, {1,5}, {3,5},H}, $\overline{\text{is}}$(H)={ ∅, {1}, {5}, {1,3}, {1,5}, {3,5},H},the map $\text{f}: (\text{H},\tau) \rightarrow (\text{K},\sigma)$ is difine as:$\text{f}(1) = 6, \text{f}(3) = (9), \text{f}(5) = 3$. Here is f is $\widehat{\text{gis}}$-continuous but is not $\overline{\text{is}}$-continuous, sg-comtinuous, gα-continuous, and rg-continuous mappings, because $\text{f}^{-1}\{9\} = \{3\}$ is only gis-open set.

Remark 3.11 We can proof easily in the same way that continous (resp.semi-coninuous, α-continuouos, regular-continuous) mappings are $\widehat{\text{gis}}$-continuous mapping.

Definition 3.12 The Mapping$f: \tau_o \rightarrow \sigma_0$ is $\widehat{gis}$-irresolute, if $\forall V \subseteq \sigma_{0\widehat{gis}}, f^{-1}(V) \subseteq \tau_{o\widehat{gis}}$.

Example 3.13 Let Q=W={s,r,t}, $\tau = \{\emptyset, \{t\}, Q\}, \sigma = \{\emptyset, \{t\}, W\}$, $\widehat{gis}$ (Q)= { ∅,{s}, {r},{t}, {s,r}, {s,t}, {r,t},Q}, $\widehat{gis}$ (W)= { ∅,{s}, {r},{t}, {s,r}, {s,t}, {r,t},w}$f: (Q, \tau) \rightarrow (W, \sigma)$ is difine as : f(r)=s, f(s)=t, f(t)=r. It is clear the mapping is $\widehat{gis}$-irresolute mapping.

Theorem 3.14 If f is α (resp. rg, and sg)-irresolute mapping, then f is $\widehat{gis}$-irresolute mapping.

Proof. Let $f: \tau_o \rightarrow \sigma_0$, be an α(resp. rg, and sg)-irresolute, and H any α(resp. rg, and sg)-open subset of σ_0, hence H is $\widehat{gis}$-open by (**Corollary2.7, Theorem 2.17** and **Corollary 2.11**). Since f is an α-irresolute, then $f^{-1}(H)$ is α-open inτ_o, and since every α(resp. rg, and sg) -open set is $\widehat{gis}$-open set. Therefore f is $\widehat{gis}$-irresolute mapping■

Theorem 3.15 Every $\widehat{gis}$-irresolute is $\widehat{gis}$-continuous mapping.

Proof. Clear form **Definition 3.12**■

Theorem 3.16 if $f: \tau_o \rightarrow \sigma_0$, and $g: \sigma_0 \rightarrow Z_0$, then $f \circ g: \tau_o \rightarrow Z_0$ is $\widehat{gis}$-irresolute.

Proof. Let R is giα-open set in Z_0, since $g: \sigma_0 \rightarrow Z_0$ is $\widehat{gis}$-irresolute, then $g^{-1}(R)$ is $\widehat{gis}$-open set inσ_0 , since$f: \tau_o \rightarrow \sigma_0$ is $\widehat{gis}$-irresolute, it imply $f^{-1}(g^{-1}(R))$ is $\widehat{gis}$-open set in τ_o. Therefore $f \circ g: \tau_o \rightarrow Z_0$ is $\widehat{gis}$-irresolute■

Corollary 3.17 if $g: \sigma \rightarrow Z_0$ is $\widehat{gis}$ – irresolute, and $f: \tau_o \rightarrow \sigma_0$ is $\widehat{gis}$-continuous mapping , prove $f \circ g$ is $\widehat{gis}$-irresolute.

Proof. Same as the proof above **Theorem 3.16**■

Definition 3.18 The Mapping $f: \tau_o \rightarrow \sigma_0$ is $\widehat{gis}$-open, if $\forall N \subseteq \tau_o, f(N) \subseteq \sigma_0$.

Example 3.19 Let B={5,7,9} ,C={4,6,8,}, $\tau = \{\emptyset\{9\}, \{7,9\}, B\}$, $\sigma = \{\emptyset, \{8\}, C\}$, $\widehat{gis}$ (C)= { ∅, {4}, {6},{8}, {4,6}, {4,8}, {6,8} ,C}$f: (B, \tau) \rightarrow (C, \sigma)$ is difine as: f(5)=8, f(7)=4, f(9)=6. The mapping is $\widehat{gis}$-open mapping.

Theorem 3.20 If f is $\overline{is}$-open mapping, then f is $\widehat{gis}$-open mapping.

Proof. Let $f: \tau_o \rightarrow \sigma_0$ is an $\overline{is}$-open function, and A is any open in τ_o, then $f(A)$ is an $\widehat{gis}$-open in σ, since every $\overline{is}$-open is $\widehat{gis}$-open by **Lemma 2.4**, so $f(A)$ is $\widehat{gis}$-open. Therefore f is $\widehat{gis}$-open■

Theorem 3.21 if f rg-open mapping, then f is $\widehat{gis}$-open.

Proof. Same the proof the above theorem by using **Theorem 2.17**■

The converse of **Theorems 3.20, and 3.21** are not true■

Example 3.22 Let B= {1,2,3}, C={4,5,6},$\tau = \{\emptyset, \{2\}, B\}$, $\sigma = \{\emptyset, \{4,6\}, C\}$, $\widehat{gis}$ (C)= { ∅,{4}, {6},{8}, {4,6}, {4,8},{6,8},C}. The identity mapping $f: (B, \tau) \rightarrow (C, \sigma)$ is $\widehat{gis}$-open mapping but it is not iα-open and rg-open mapping, because $f\{2\} = \{2\}$ is not $\overline{is}$-open and rg-open.

Theorem 3.23 If $: \tau_o \rightarrow \sigma_0$ is open mapping, and $g: \sigma_0 \rightarrow Z_0$ is $\widehat{gis}$-open mapping, then the composition $f \circ g$ is $\widehat{gis}$-open map.

Proof. Let $H \subset \tau_o$, snice $f: \tau_o \rightarrow \sigma_0$ is open map, then $f(H) \subset \sigma_0$, also since $g; \sigma_0 \rightarrow Z_0$

is $\widehat{gis}$-open map, then $g\big(f(H)\big) \subset Z_0$ is $\widehat{gis}$-open. Therefore $f \circ g$ is $\widehat{gis}$-open map■

Remark 3.24

i) If f and g are $\widehat{gis}$-open map, then $f \circ g$ is not $\widehat{gis}$-open map.

ii) The proof of the above **Theorems 3.20** and **3.21**are true for $\widehat{gis}$-closed mapping.

iii) The **Diagram 2.24** is true for $\widehat{gis}$-open (resp. continuous, and irresolute) mappings

Theorem 3.25 The bijection mapping$f: \tau_o \to \sigma_0$, these are equivalent statements as the following.

(i) $f^{-1}: \tau_o \to \sigma_0$ is $\widehat{gis}$- continuous.

(ii) f is $\widehat{gis}$-open map.

(iii) f is $\widehat{gis}$- closed map.

Proof. **(i) ⟹(ii)** Let $M \subset \tau_o$. Since f is $\widehat{gis}$ −continuous mapping, ${f^{-1}}^{-1}$(M) $=f(M) \subset \sigma_{0\widehat{gis}}$. Therefore f is $\widehat{gis}$-open map■

(ii) ⟹ (iii) Let $N \subset \overline{\tau_o}$, then $N^c$is $\subset \tau_o$. Since f is $\widehat{gis}$ −open mapping, $f(N^c) \subset \sigma_{0\widehat{gis}}$. Hence $f(N) \subset \tau_{o\widehat{gis}}$. Therefore f is $\widehat{gis}$-closed mapping■

(iii) ⟹ (i) Let R$\subset \tau_{o\widehat{gis}}$, Since f is $\widehat{gis}$ −closed mapping, then $f(R) \subset \overline{\sigma_0}_{\widehat{gis}}$. henever $f(R) = {f^{-1}}^{-1}(R)$. Hence $f^{-1}: \tau_o \to \sigma_0$ is $\widehat{gis}$- continuous■

Definition 3.26 A bijection giα –continuous, and giα-open mappings (resp. f and f^{-1} is $\widehat{gis}$- irresolute) is called $\widehat{gis}$–homeomorphism (resp. $*\widehat{gis}$–homeomorphism).

Theorem 3.27 If f is rg-homeomorphism, then f is $\widehat{gis}$–homeomorphism.

Proof. Let $f: \tau_o \to \sigma_0$ is rg-homeomorphism. Hence f is $\widehat{gis}$–continuous because every gr-continuous is $\widehat{gis}$ by **Theorem 3.9**. In addition, by using **Theorem 21,** then f $\widehat{gis}$-open mapping. Therefore f is $\widehat{gis}$-homeomorphism■

Theorem 3.28 If f is $\overline{is}$-homeomorphism, hence f is $\widehat{gis}$–homeomorphism.

Proof. Same the proof of above theorem, but by using **Theorem 3.7**, and **Theorem 3.20** ■

Remark3.29 Example 3.10 is enough to proof that the converse of **Theorem 3.27** is not always true.

$\widehat{gis}$-Separation axioms with its applications

Gradually this section presents some new weak of $\widehat{gis}$ −separation axioms with theoretical result.

Definition 4.1 Let (X, τ_o) topology space, then (X, τ_o) is defined to be:

(i) $\widehat{gis}$-*T₀*, if $\forall\, n \neq m \in X, \exists\, B \subseteq \tau_{\widehat{gis}}$, s.t $n \in B \wedge m \notin B$.

(ii) $\widehat{gis}$-T_1, if $\forall\, n \neq m \in X$, $\exists\, B \wedge C \subseteq \tau_{\widehat{gis}}$, s.t $n \in B \wedge m \notin B, m \in C \wedge n \notin C$.

(iii) $\widehat{gis}$-T_2, if $\forall\, n \neq m \in X$, $\exists\, B \neq C \subseteq \tau_{\widehat{gis}}$, s.t $n \in B \wedge, m \in C$.

(iv) $\widehat{gis}$–regular, if $\forall\, n \notin \lambda \subseteq \overline{\tau_o}$, $\exists\, B \neq C \subseteq \tau_{\widehat{gis}}$, s.t $n \in B \wedge \lambda \subseteq C$.

(v) $\widehat{gis}$–normal, if $\forall\, \nu \subseteq \overline{\tau_o} \neq \lambda \subseteq \overline{\tau_o}$, $\exists\, B \neq C \subseteq \tau_{\widehat{gis}}$, s.t $\nu \subseteq B \wedge \lambda \subseteq C$.

(vi) $\overline{is}$-$T_{1/2}$, if $\overline{\tau_{\overline{is}}} = \overline{\tau_{\iota}}$.

Example 4.2 $X = \{\alpha, \beta, \gamma\}$,$\tau = \{\emptyset, \alpha, X\}$, $SO(X) = \{\emptyset, \{\alpha\}, \{\beta, \alpha\}, \{\alpha, \gamma\}, X\} \subset \overline{is} = \widehat{gis} = \{\emptyset,$ $\{\alpha\}, \{\beta\}, \{\gamma\}, \{\beta, \alpha\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, X\}$.

Clearly, τ is $\widehat{gis}$-T_o, $\widehat{gis}$-T_1, $\widehat{gis}$-T_2, $\widehat{gis}$-regular, $\widehat{gis}$-normal, and $\widehat{gis}$-$T_{1/2}$.

Theorem 4.3 Every $\overline{is}$-T_o (resp. $\overline{is}$-T_1, $\overline{is}$-T_2, $\overline{is}$-*regular,* $\overline{is}$-*normal,* $\overline{is}$-$T_{1/2}$) are $\widehat{gis}$-T_o(resp. $\widehat{gis}$-T_1, $\widehat{gis}$-T_2, $\widehat{gis}$-*regular,* $\widehat{gis}$-*normal,* $\widehat{gis}$-$T_{1/2}$).

Proof. It is evident by using **Lemma 2.4**■

Theorem 4.4 X is a $\widehat{gis}$-$T_{1/2}$ *iff* $\forall\, x \in X$, {x} is $\widehat{gis}$-open or $\widehat{gis}$-closed in $\widehat{gis}$-$T_{1/2}$, i.e., X is a $\widehat{gis}$-$T_{1/2}$ *iff* a τ_o is a $\widehat{gis}$-$T_{1/2}$.
Proof. It obvious from **Definition $\widehat{gis}$-$T_{1/2}$**■

Theorem 4.5 if $\tau \subset \tau_s$, then τ is $\widehat{gis}$-$T_{1/2}$.
Proof. Resemble the proof of **Theorem 2.21**■
Theorem 4.6 if $f: \tau_o \to \sigma_0$ is injection $\widehat{gis}$-open mapping, and τ_o is $\widehat{gis}$-$T_{1/2}$, then σ_0 is $\widehat{gis}$-$T_{1/2}$.

Theorem 4.7 if $f: \tau_o \to \sigma_0$ is injection $\widehat{gis}$-irresolute mapping, and σ_0 $\widehat{gis}$-$T_{1/2}$, then τ_o is $\widehat{gis}$-$T_{1/2}$.

Theorem 4.8 If $\tau_o = \sigma_0$ are both $\widehat{gis}$-$T_{1/2}$, and f is injection, then $f: \tau_o \to \sigma_0$ is:

(i) $\widehat{gis}$-open ⇒(ii)$\widehat{gis}$-continuous ⇒ (ii) $\widehat{gis}$-irresolute mappings.

Proof (i). Let $G \neq Q$ are open sets in τ_o, since f is injection and $\tau_o = \sigma_0$,then $f(G) \neq f(Q)$ are open in σ_0. Since σ_0 is $\widehat{gis}$-$T_{1/2}$, it implies $\sigma_0 \subset \sigma_{\widehat{gis}}$, hence $f(G) \neq f(Q)$ are $\widehat{gis}$-open set in σ_0. Therefore, f is $\widehat{gis}$-open mapping■
Proof (ii). Let $G \neq Q$ are open sets in σ_0, since $\tau_o$is $\widehat{gis}$-$T_{1/2}$ f is injection $\widehat{gis}$-open mapping, then $f(G)^{-1} \neq f(Q)^{-1}$ are $\widehat{gis}$-open sets in τ_o. Therefore, f is $\widehat{gis}$-continuous mappings■

Proof (iii). It is obvious by **Theorem 3.15** ■

Corollary 4.9 If $\tau_o = \sigma_0$ are both $\widehat{gis}$-$T_{1/2}$, and f is injection, then $f: \tau_o \to \sigma_0$ is$\widehat{gis}$-homeomorphism.

Proof. It is obvious by **Definition 3.22(i)**

Corollary 4.10 If $\tau_o = \sigma_0$ are both $\widehat{gis}$-$T_{1/2}$, and f is injection, then$f: \tau_o \to \sigma_0$, and $f^{-1}: \tau_o \to \sigma_0$ are $\widehat{gis}$-irresolute mappings.

Corollary 4.11 If $\tau_o = \sigma_0$ are both $\widehat{gis}$-$T_{1/2}$, and f is injection, then $f: \tau_o \to \sigma_0$ is *$\widehat{gis}$-homeomorphism.

Proof. It is obvious by **definition 3.22(ii)**

Theorem 4.12 If $\mathrm{f}: \tau_o \to \sigma_0$ is injection $\widehat{gis}$-irresolute, and σ_0 is $\widehat{gis}$-T_o, then τ_o $\widehat{gis}$-T_2.

Proof. Suppose $g \neq q \in \tau_o$ and f be an injective, then $f(g) \neq f(q)$ in σ_0. Since σ_0 is an $\widehat{gis}$-T_o, $\exists N \subseteq \tau_{\widehat{gis}}$, s.t $f(g) \in N$, $\wedge$ $f(q) \notin N$, ⇒ $g \in f^{-1}(N)$ $\wedge$ $q \notin f^{-1}(N)$. Since, $f$is $\widehat{gis}$- irresolute, then $f^{-1}(M) \subset \tau_{\widehat{gis}}$, then $g \in f^{-1}(N) \neq q \in f^{-1}(M)$.Therefore, τ_o is $\widehat{gis}$-T_2■

Theorem 4.13 If $f: \tau_o \to \sigma_0$ be a $\widehat{gis}$-continuous closed injection. If σ_0 is $\widehat{gis}$–normal, then τ_o is $\widehat{gis}$–normal.

Proof. Let $L_1 \subseteq \tau_o \neq L_2 \subseteq \tau_o$. Since f is injective and closed, $f(L_1) \subseteq \overline{\sigma_0} \neq f(L_2) \subseteq \overline{\sigma_0}$. Since σ_0 is $\widehat{\mathrm{gis}}$–normal , then $f(L_1) \neq f(L_2)$, $\exists$ $O_1 \subseteq \sigma_{\widehat{gis}} \neq O_2 \subseteq \sigma_{\widehat{gis}}$, s.t $f(L_1) \subset O_1 \neq f(L_2) \subset O_2$, we obtain$L_1 \subset f^{-1}(O_1) \wedge L_2 \subset f^{-1}(O_2)$. Since, f is an $\widehat{gis}$-continuous, then$f^{-1}(V_1) \wedge f^{-1}(V_2) \subseteq \tau_{\widehat{gis}}$. Also,$f^{-1}(L_1) \cap f^{-1}(L_2) = f^{-1}(L_1 \cap L_2) = \emptyset$. Therefore, X is an $\widehat{gis}$–normal ■

Corollary 4.14 if $f: \tau_o \rightarrow \sigma_0$ be a $\widehat{gis}$-irresolute closed, and injection. If σ_0 is $\widehat{gis}$–normal, then τ_o is $\widehat{gis}$–normal.

Proof. Since every $\widehat{gis}$-continuous is $\widehat{gis}$-irresolute mapping by **Theorem 3.11**■

Corollary 4.15 If $f: \tau_o \rightarrow \sigma_0$ be a $\widehat{gis}$-irresolute closed injection, and σ_0 is $\widehat{gis}$–regular, then τ_o is $\widehat{gis}$–regular.

Proof. It is obvious from definition of $\widehat{gis}$-regular■

CONCLUSIONS AND SUGGESTIONS

The new $\widehat{gis}$-closed and open set that mention in this paper investigated many properties for about the same type and the $\widehat{gis}$-mappings. We can expand about this new $\widehat{gis}$-set by related with other open-closed-generalized closed sets. We also present $\widehat{gis}$- lower separation axioms. In addition, we related this $\widehat{gis}$-separation with advanced $\widehat{gis}$-mappings to reach high-expended theorems. Finally, we do not forget the ability to create other type of generation closed and open sets by using other kinds of open and closed accompanied its applications.

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