Contents lists available at DJM
DESIMAL: JURNAL MATEMATIKA
p-ISSN: 2613-9073 (print), e-ISSN: 2613-9081 (online), DOI 10.24042/djm
http://ejournal.radenintan.ac.id/index.php/desimal/index

Analysis of stability and bifurcation in logistics models with harvesting in the form of the holling type III functional response

Yuni Yulida[a*], Firman Nurrobi, Faisal , Muhammad Ahsar K

Universitas Lambung Mangkurat, Indonesia
[a] Kelompok Keahlian Matematika Lingkungan dan Industri

ARTICLE INFO

Article History
Received : 14-03-2022
Revised : 20-03-2022
Accepted : 11-04-2022
Published : 30-04-2022

Keywords:
Logistic Model, Harvesting, Holling Type III, Equilibrium, Stability Analysis, Fold Bifurcation.

*Correspondence: E-mail:
y_yulida@ulm.ac.id

Doi:
10.24042/djm.v5i1.11828

ABSTRACT

The logistic model can be applied in the field of biological studies to investigate population growth problems and some important aspects of the ecological situation. This model is a growth model with a limited population growth rate, and ecologists describe this rate as carrying capacity. Carrying capacity can be interpreted as the ideal population size, where individuals in the population can live properly in their environment. The growth rate of a population can be influenced by the harvesting factor, in this case, it is assumed that harvesting is not constant. The effect of the harvest on the growth rate can be analyzed mathematically by using the Holling type III functional response. In this paper, describe the formation of a logistic model taking into account the effects of harvesting, using the Holling type III functional response. Then, perform a nondimensional process in the model, namely simplifying a model that has four parameters to a model that only has two parameters. Next, determine the equilibrium point of the model, perform a stability analysis at that equilibrium point, and investigate the possibility of bifurcation. As result, first obtained a logistic model which has two non-dimensional parameters, where one of the equilibrium points is zero and is unstable. Next, determine another equilibrium point through an implicit equation and investigate its stability through simulation. Finally, obtained two equilibrium points, which are fold bifurcation.

http://ejournal.radenintan.ac.id/index.php/desimal/index

INTRODUCTION

Problems in the field of biological mathematics regarding growth rates can be represented in the form of mathematical models. According to Paul et al. (2016), the mathematical model of population growth was built to provide some important aspects of the actual ecological situation. One of the mathematical models that discusses the problem of population growth is the exponential model and the logistic model.

The exponential model is a model that describes population growth which is only affected by births and deaths of the population. The model assumes unlimited resources and constant population birth and death rates. Thus, each population will continue to grow so that the number is getting bigger and infinite. This model does not accommodate intraspecific competition for environmental resources (Tsoularis et al., 2001). It becomes less logical because as the population becomes denser, each individual gets a smaller share of the resource. This should cause every individual in a population to experience competition (either food, shelter, or other resources) to maintain their life in the environment (Karim & Yulida, 2019). This model is called the logistic model which is a model with a limited population growth rate. Ecologists define it in terms of carrying capacity as the ideal population size so that individuals in the population can live properly in an environment (Rahmi & Panigoro, 2017).

Furthermore, another factor that can affect the growth rate is the harvesting factor. This can happen because the population grows and begins to interact with the environment or predators of that population. Harvesting factors can be in the form of constant harvesting and non-constant harvesting. Harvesting itself is the process of taking a number of populations per unit time. Constant harvesting is harvesting that has a fixed value regardless of the population being harvested. While harvesting is not constant, the harvesting depends on the number of the harvested population.

Related studies are Karim & Yulida (2019) which use logistic models to estimate parameters and predict the population of South Kalimantan, Doust & Saraj (2015), Hidayati (2018), and Nurrobi et al. (2017) which discuss the logistic model with constant and non-constant harvesting factors using the type II Holling response function and analyzing through the solution and model stability.

Other studies that use the Holling function in the predation process include Etoua & Rousseau (2010), which propose an expanded Gause model and uses an expanded type III Holling response function in the predation process and uses a constant harvesting factor for prey populations. Then the research of Maziun & Subchan (2020), used the Holling type III response function in the predation process for a predatory prey system with delayed stones. Furthermore, a study conducted by Chen (2022), which used harvesting factors in prey populations by following the Holling type II functional response, analyzed the stability and investigated the bifurcation that occurred.

In this paper, the authors are interested in explaining the formation of a logistic model with harvesting using the Holling type III functional response, determining the equilibrium point of the model, analyzing the stability of the equilibrium point, and performing simulations. In the simulation, it is presented the possible equilibrium points are formed and each is investigated for stability analysis and investigates the occurrence of bifurcation.

METHOD

This research begins by explaining the process of forming a harvesting factor in the form of the Holling type III function in the logistics model. The model contains four parameters so that it is carried out non-dimensionally into a model with two parameters, determines the equilibrium point of the model, analyzes the stability of the equilibrium point, and performs simulations. The simulation presents the possible equilibrium points that occur and each is investigated for stability analysis by taking into account two parameters. Finally, a bifurcation diagram that might occur in the model is presented.

The theory related to the model stability analysis process is presented as follows :

Stability of a first-order differential equation

An ordinary differential equation of first order is a differential equation that contains the highest derivative of one. According to Ndii (2018), the form of a first-order ordinary differential equation is $\frac{dx}{dt} = f(x)$

The equilibrium point of the equation $\frac{dx}{dt} = f(x)$, if $f(\hat{x}) = 0$ (Hale & Kocak, 1991). If $f(x)$ is approximated linearly using a Taylor series around the point $\hat{x}$ then it can be expressed as $\frac{dx}{dt} = f'(\hat{x})(x - \hat{x})$, where $\lambda = f'(\hat{x}) = \frac{df(x)}{dx}\Big|_{x=\hat{x}}$ is the eigenvalue of the linearized equation (Martcheva, 2015).

While the stability analysis around the equilibrium point $\hat{x}$ is based on the eigenvalues according to Martcheva (2015), Olver & Shikiban (2003), and Hale & Kocak (1991) is $\hat{x}$ asymptotically stable if $f'(\hat{x}) < 0$, and $\hat{x}$ is unstable if $f'(\hat{x}) > 0$.

Furthermore, according to Kuznetsov (1998), bifurcation is the emergence of a different dynamic state of a system due to changes in parameters. If the eigenvalue obtained is zero, then this bifurcation is called a fold or saddle-node bifurcation.

Logistics model

According to Cain & Reynolds (2010) and Yulida & Karim (2021), the logistics growth model can be written as follows.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Meanwhile, according to Doust & Saraj (2015), explaining that the logistic model with harvesting can be written as follows.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - h(t)$$

where $N(t)$ represents the total population at time t, r represents the intrinsic growth rate, K is the carrying capacity, and h is the harvesting function.

RESULTS AND DISCUSSION

One form of inconstant harvesting is harvesting in the form of the Holling type III function introduced by Ludwig et al. (1978) in a mathematical model for a population of spruce budworms. Ludwig's model describes the population of caterpillars that live in trees with carrying capacity in the form of the number of available leaves, which are then eaten or harvested by bird predators, where the total population of caterpillars that are eaten or harvested follows the Holling type III function (Robeva & Murrugarra, 2016).

The logistic model presented in this paper involves a harvesting factor in the form of the Holling type III function. The Holling type III function assumes that predators tend to look for other populations when the density of their prey population begins to decrease. Let Y be the number of prey populations that will be consumed by one predator, can be expressed as $Y = a^* T_s N$, where a^* is the attack rate or search efficiency carried out in an interval of time to search for prey T_s against a number of prey populations N. Furthermore, it is assumed that predators pay attention to the number of populations they will prey on, as a result, predators will meet more of their prey. This causes the rate of capture to depend on the number of prey populations, namely $a^* = aN$ where a is the catch rate per prey population. So that the number of prey population that will be consumed by one predator becomes $Y = aT_s N^2$.

In addition to time to find prey, predators need time to eat their prey. The total time (T_t) required by the predator is $T_t = T_s + T_h = T_s + bY$, where $T_h = bY$ is the handling time which depends on the number of prey population consumed by the predator and b is the handling time required by the predator per prey.

Time to search for prey will decrease because there is time to handle prey, so the number of prey populations consumed by predators will increase $Y = a(T_t - bY)N^2$ or $Y = \frac{T_t}{b}\frac{N^2}{\frac{1}{ab}+N^2}$ if both sides are divided by T_t, then the number of prey population consumed per total time spent by predators $(\frac{Y}{T_t})$ is

$$\frac{Y}{T_t} = \frac{1}{b}\frac{N^2}{\frac{1}{ab} + N^2} \quad (1)$$

Suppose $\beta = \frac{1}{b}$ and $\alpha^2 = \frac{1}{ab}$, where β is the number of prey populations handled by predators per time which is also known as predation rate by predators, α represents the number of prey populations that are the threshold for changing predation. So, Equation (1), where $\frac{Y}{T_t} = g(N)$ represents the Holling type III function which can be written as follows

$$g(N) = \beta\,\frac{N^2}{\alpha^2 + N^2} \quad (2)$$

Based on the explanation above, it can be presented a logistic model with harvesting in the form of the Holling type III functional response, as follows:

$$\frac{dN}{dt} = rN\left(\frac{K-N}{K}\right) - \beta\,\frac{N^2}{\alpha^2 + N^2} \quad (3)$$

The stages in analyzing the stability of Equation (3) are

1. Before determining the equilibrium point, Equation (3) can be transformed into a non-dimensional equation. Determine the point that satisfies (the right-hand side of the non-dimensional equation is zero).
2. Determining the eigenvalues through the characteristic equation of the Jacobian matrix, in this case, because the model only consists of one equation, the Jacobian matrix can be determined through the first derivative for the right-hand side of the non-dimensional equation with respect to the dependent variable.
3. Furthermore, to find out whether there is a bifurcation, it can be seen through changes in parameters that will cause changes in the number of equilibrium points and eigenvalues (changes in stability).
4. In this paper, the simulation is given for the equilibrium point which cannot be determined explicitly using parametric curves, and is continued for stability analysis and bifurcation diagrams that occur.

Equilibrium point model

In Equation (3) there are four parameters, namely r, K, α, β, to analyze the behavior of the model, nondimensional model is carried out with four parameters to become a model with two parameters. Let $N = \alpha n$ and $t = \frac{\alpha\tau}{\beta}$., using the chain rule obtained

$$\frac{dN}{dt} = \beta\frac{dn}{d\tau} \quad (4)$$

Based on the example and Equation (4), so Equation (3) can be written as

$$\frac{dn}{d\tau} = Rn\left(1 - \frac{n}{Q}\right) - \frac{n^2}{1+n^2} \quad (5)$$

where $R = \frac{\alpha r}{\beta}$ and $Q = \frac{K}{\alpha}$.

Furthermore, from Equation (5) it can be determined the equilibrium point, namely

$$Rn\left(1 - \frac{n}{Q}\right) - \frac{n^2}{1+n^2} = 0 \quad (6)$$

From Equation (6), it is obtained

$$n = \hat{n}_0 = 0 \quad (7)$$

or

$$R\left(1 - \frac{n}{Q}\right) = \frac{n}{1+n^2} \quad (8)$$

Equation (8) is written

$$y(n) = h(n) \quad (9)$$

To determine the equilibrium point that satisfies Equation (8), it can be found through the intersection of the two curves of Equation (9). To determine the value of R and Q that causes one, two, or three equilibrium points, it is determined where the $y(n)$ and $h(n)$ curves intersect and intersect. This happens if $\frac{-R}{Q} = \frac{1-n^2}{(1+n^2)^2}$.

Then R and Q will be determined that meet Equation (8), namely

$$R = \frac{-Q(1-n^2)}{(1+n^2)^2} \quad (10)$$

Equation (10) is substituted into Equation (8) and obtained

$$Q = \frac{2n^3}{(n^2-1)} \quad (11)$$

Based on Equation (11), then Equation (10) can be expressed as

$$R = \frac{2n^3}{(1+n^2)^2} \quad (12)$$

Model stability analysis

To analyze the one-equation model, based on Otto & Day (2007) and Olver & Shikiban (2003) explained that the eigenvalues can be determined using the first derivative. Based on Model (5), suppose the right-hand side is

$$G(n) = Rn\left(1-\frac{n}{Q}\right) - \frac{n^2}{1+n^2}$$

It can be determined that the first derivative of n is

$$G'(n) = R - \frac{2R}{Q}n - \frac{2n}{(1+n^2)^2} \quad (13)$$

The following is an analysis of the stability of the model around the equilibrium point with the parameters R and Q being positive constants with the following conditions:

Suppose $n = \hat{n}$ is an equilibrium point that satisfies parametric equations (11) and (12) then the first derivative of $G(n)$ with respect to n at the point $n = \hat{n}$ is $G'(\hat{n}) = 0$. Because $\lambda = G'(\hat{n}) = 0$ is an eigenvalue at the equilibrium point $n = \hat{n}$, then there is a fold bifurcation (saddle-node).

Stability analysis and equilibrium point bifurcation are described by the following simulation.

Model simulation

The following is the form of a parametric curve of the R and Q functions using Equation (11) and Equation (12). The equation is a parametric equation of n, that is, each value of n determines a pair of points (R, Q) that form a parametric curve in the RQ plane. The parametric curve can be described in Figure 1.

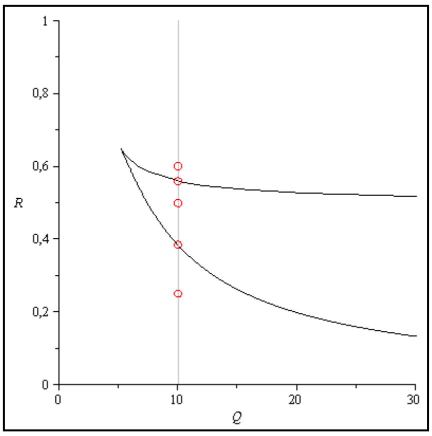

Figure 1. Parametric Curves $\boldsymbol{R(n)}$ and $\boldsymbol{Q(n)}$ on the $\boldsymbol{RQ}$ plane

Based on Figure 1, the values of R and Q will be taken to illustrate the simulation, with Q fixed and R increasing. Suppose that the value of $Q = 10.02$ (fixed), while the value of R is

1. under the curve that is $R = 0.25$
2. in the middle of the curve that is $R = 0.499$
3. For the values of R and Q on the curve

At the value of $Q = 10.02$, based on Equation (11) it becomes $2n^3 - 10n^2 + 10 = 0$ and has a solution of $n = 1.137$, $n = 4.791$ and $n = -0.919$. Because $n > 0$ then what is taken is $n = 1.137$ and $n = 4.791$, so there are only two points on the curve, namely at $n = 1.137$, that is $R = 0.559$ and at $n = 4.791$, that is $R = 0.384$

4. above the curve that is $R = 0.6$

Simulation at conditions $\boldsymbol{R = 0.25}$ and $\boldsymbol{Q = 10.02}$

Suppose the initial population is $N(0) = 10$ and $N(0) = 150$. If given the values of the parameters are $\alpha = 99{,}80$, $\beta = 200$, $K = 1000$, and $r = 0.5$. Since $n = \frac{N}{\alpha}$ the initial values in n are $n(0) = 0.1$ and $n(0) = 1.5$. Because $R = \frac{\alpha r}{\beta}$ then $R = 0.25$ and $Q = \frac{K}{\alpha}$ then $Q = 10.02$.

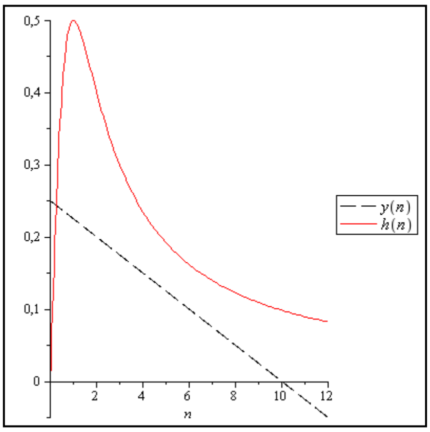

Figure 2. $\boldsymbol{h(n)}$ and $\boldsymbol{y(n)}$ Curves with $\boldsymbol{R = 0.25}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 2, it is obtained that one intersection as the equilibrium point is $\hat{n}_1 = 0.259$.

Next, Figure 3 is given, which is a model phase portrait.

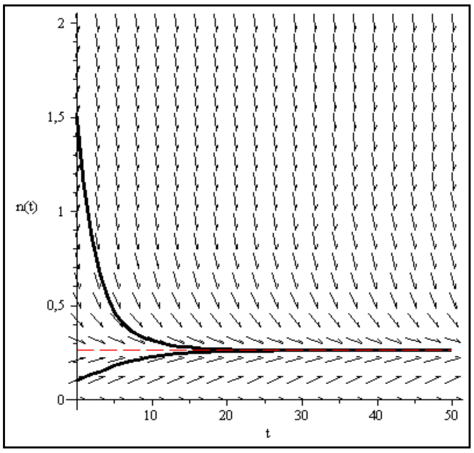

Figure 3. The Model Phase Portrait with $\boldsymbol{R = 0.25}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 3, it can be explained that

a. When the initial value is $n(0) < \hat{n}_1$ the population will increase to condition $\hat{n}_1$.
b. When the initial value is $n(0) > \hat{n}_1$ the population will decrease towards the condition $\hat{n}_1$.

Based on the analysis above, the equilibrium point $\hat{n}_1$ is asymptotically stable.

Simulation at conditions $\boldsymbol{R = 0.384}$ and $\boldsymbol{Q = 10.02}$

Suppose the initial population are $N(0) = 10$, $N(0) = 150$ and $N(0) = 600$. If given the value of the parameter, namely $\alpha = 99.80$, $\beta = 130$, $K = 1000$, and $r = 0.5$. Since $n = \frac{N}{\alpha}$ the initial values in n are $n(0) = 0.1$, $n(0) = 1.5$, and $n(0) = 6$. Because $R = \frac{\alpha r}{\beta}$ then $R = 0.384$ and $Q = \frac{K}{\alpha}$ then $Q = 10.02$.

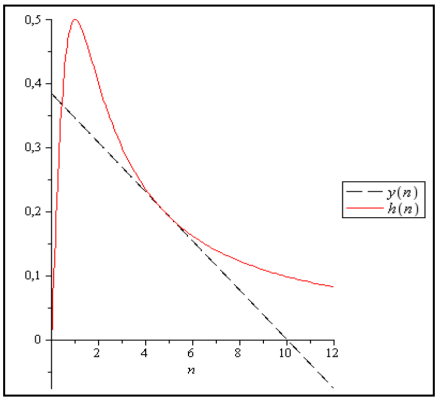

Figure 4. $\boldsymbol{h(n)}$ and $\boldsymbol{y(n)}$ Curves with $\boldsymbol{R = 0.384}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 4, two intersections are obtained and the equilibrium point is obtained, namely $\hat{n}_1 = 0.437$ and $\hat{n}_2 = 4.791$.

Next, Figure 5 is given, which is a model phase portrait.

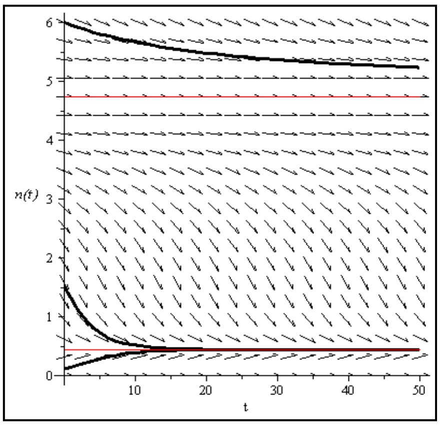

Figure 5. The Model Phase Portrait with $\boldsymbol{R = 0.384}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 5, it can be explained that

a. when $n(0) < \hat{n}_1$ the population will increase closer $\hat{n}_1$.
b. when $\hat{n}_1 < n(0) < \hat{n}_2$ the population will increase towards $\hat{n}_1$ and away from $\hat{n}_2$.

c. when $n(0) > \hat{n}_2$ the population will decrease towards $\hat{n}_2$.

Based on the analysis above, point $\hat{n}_1$ is asymptotically stable and point $\hat{n}_2$ is semi-stable.

Simulation at conditions $\boldsymbol{R = 0.499}$ and $\boldsymbol{Q = 10.02}$

Suppose the initial population are $N(0) = 10$, $N(0) = 150$, $N(0) = 600$, dan $N(0) = 800$. If given the value of the parameters, namely $\alpha = 99.80$, $\beta = 100$, $K = 1000$, and $r = 0.5$. Since $n = \frac{N}{\alpha}$ the initial values in n are $n(0) = 0.1$, $n(0) = 1.5$, $n(0) = 6$, and $n(0) = 8$. Because $R = \frac{\alpha r}{\beta}$ then $R = 0.499$ and $Q = \frac{K}{\alpha}$ then $Q = 10.02$.

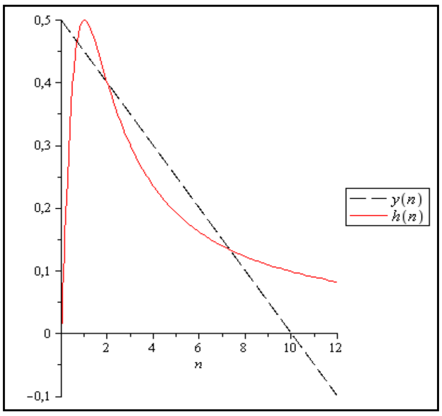

Figure 6. $\boldsymbol{h(n)}$ and $\boldsymbol{y(n)}$ Curves with $\boldsymbol{R = 0.499}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 6, three intersections are obtained and the equilibrium point is obtained, namely $\hat{n}_1 = 0.680$, $\hat{n}_2 = 2.008$, and $\hat{n}_3 = 7.331$.

Next, Figure 7 is given, which is a model phase portrait.

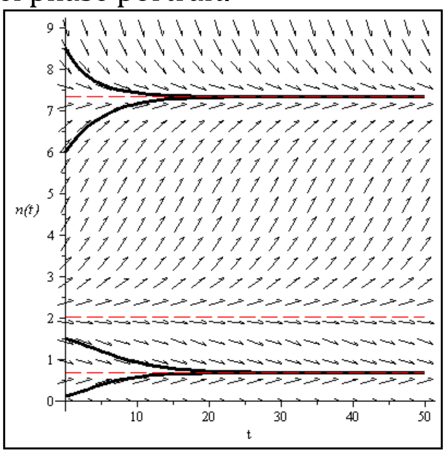

Figure 7. The Model Phase Portrait with $\boldsymbol{R = 0.499}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 7, it can be explained that

a. when $n(0) < \hat{n}_1$ the population will increase closer to $\hat{n}_1$.
b. when $\hat{n}_1 < n(0) < \hat{n}_2$ the population will increase towards $\hat{n}_1$ and away from $\hat{n}_2$.
c. when $\hat{n}_2 < n(0) < \hat{n}_3$ the population will increase towards $\hat{n}_3$ and away from $\hat{n}_2$.
d. when $n(0) > \hat{n}_3$ the population will decrease towards $\hat{n}_3$.

Based on the above analysis, points $\hat{n}_1$ and $\hat{n}_3$ are asymptotically stable and point $\hat{n}_2$ is unstable.

Simulation at conditions $\boldsymbol{R = 0.559}$ and $\boldsymbol{Q = 10.02}$

Suppose the initial population are $N(0) = 10$, $N(0) = 150$, $N(0) = 600$, and $N(0) = 800$. If given the value of the parameters, namely $\alpha = 100$, $\beta = 89.44$, $K = 1000$, and $r = 0.5$. Since $n = \frac{N}{\alpha}$ the initial values in n are $n(0) = 0.1$, $n(0) = 1.5$, $n(0) = 6$, and $n(0) = 8$. Then the value of $R = 0.559$ and $Q = 10.02$.

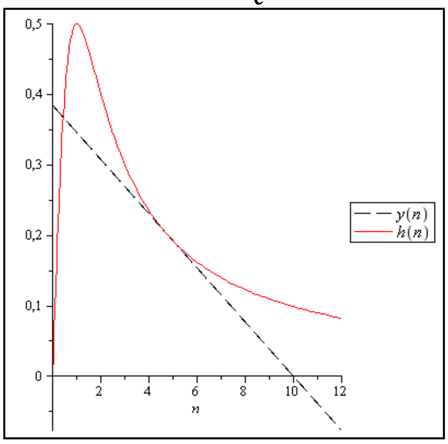

Figure 8. $\boldsymbol{h(n)}$ and $\boldsymbol{y(n)}$ Curves with $\boldsymbol{R = 0.559}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 8, two intersections are obtained and the equilibrium point is obtained, namely $\hat{n}_1 = 1.137$ and $\hat{n}_2 = 7.743$.

Next, Figure 9 is given, which is a model phase portrait.

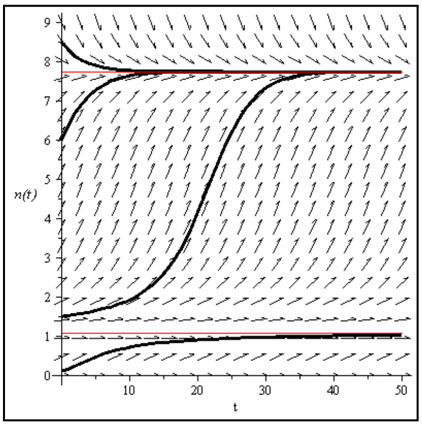

Figure 9. The Model Phase Portrait with $\boldsymbol{R = 0.559}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 9, it can be explained that

a. when $n(0) < \hat{n}_1$ the population will increase closer to $\hat{n}_1$.
b. when $\hat{n}_1 < n(0) < \hat{n}_2$ the population will increase towards $\hat{n}_2$ and away from $\hat{n}_1$.
c. when $n(0) > \hat{n}_2$ the population will decrease towards $\hat{n}_2$.

Based on the above analysis, point $\hat{n}_2$ is asymptotically stable and the equilibrium point $\hat{n}_1$ is semi-stable.

Simulation at conditions $\boldsymbol{R = 6}$ and $\boldsymbol{Q = 10.02}$

Suppose the initial population are $N(0) = 10$, $N(0) = 150$, $N(0) = 600$, and $N(0) = 800$. If given the value of the parameters, namely $\alpha = 99.80$, $\beta = 83.33$, $K = 1000$, and $r = 0.5$. Since $n = \frac{N}{\alpha}$ the initial values in n are $n(0) = 0.1$, $n(0) = 1.5$, $n(0) = 6$, and $n(0) = 8$. Then the value of $R = 0.6$ and $Q = 10.02$.

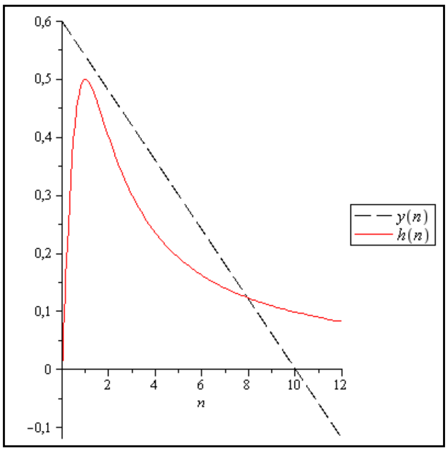

Figure 10. $\boldsymbol{h(n)}$ and $\boldsymbol{y(n)}$ Curves with $\boldsymbol{R = 0.6}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 10, one intersection is obtained and the equilibrium point is obtained, namely $\hat{n}_1 = 7.952$.

Next, Figure 11 is given, which is a model phase portrait.

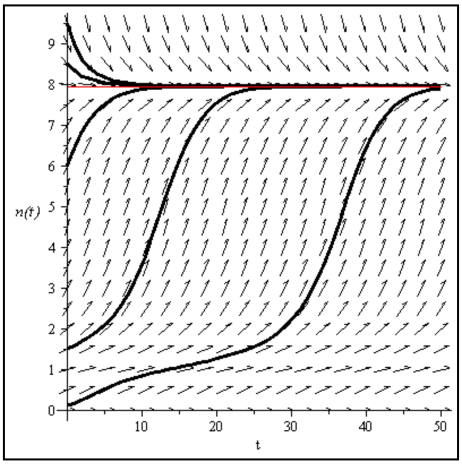

Figure 11. The Model Phase Portrait with $\boldsymbol{R = 0.6}$ and $\boldsymbol{Q = 10.02}$

Based on Figure 11, it can be explained that

a. when $n(0) < \hat{n}_1$ the population will increase closer to $\hat{n}_1$.
b. when $n(0) > \hat{n}_1$ the population will decrease towards $\hat{n}_1$.

Model bifurcation diagram

Based on the simulation (in the form of approximate values) in the condition, $(R, Q) = (0.384, 10.02)$ produces an equilibrium point $\hat{n} = 4.791$ and $(R, Q) = (0.559, 10.02)$ produces an equilibrium point n $\hat{n} = 1.137$. The two equilibrium points are semi-stable. Then, proceed with the investigation through the bifurcation diagram. When $Q = 10.02$, then Equation (6) becomes

$$\frac{dn}{d\tau} = Rn\left(1 - \frac{n}{10.02}\right) - \frac{n^2}{1 + n^2} = G(n, R) \quad (14)$$

Furthermore, the eigenvalues of Model (14) are

$$\lambda = G'(\hat{n}, R) \quad (15)$$

For the condition $\hat{n} = 4.791$, $\lambda = G'(4.791, 0.384) \approx 0$ is obtained. For $\hat{n} = 1.137$, $\lambda = G'(1.137, 0.559) \approx 0$ is obtained. Based on this, the equilibrium point experiences a fold bifurcation (saddle-node). Next, a bifurcation diagram in the (n, R). plane is given. For example, in Equation (14), $G(n, R) = 0$, then the model bifurcation diagram can be presented in Figure 12.

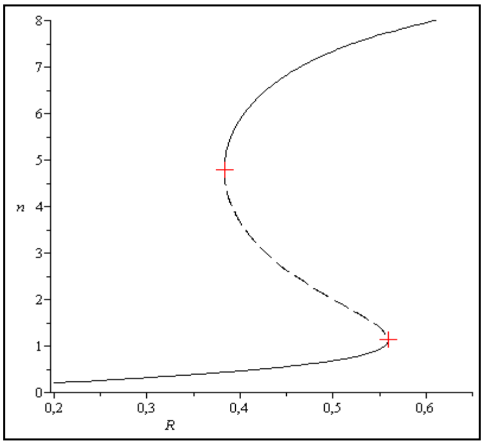

Figure 12. Model Bifurcation Diagram with $\boldsymbol{Q = 10.02}$

In Figure 12, based on the simulations performed, the equilibrium point between $1.137 < n < 4.791$ is unstable and the equilibrium point at $0 < n < 1.137$ or $n > 4.791$ is stable.

CONCLUSIONS AND SUGGESTIONS

The logistic model with harvesting in the form of the Holling type III functional response has four parameters. To analyze the behavior of the model can be done by non-dimensionalizing the model into two parameters. Based on this model, it is found that the equilibrium point is equal to unstable zero, while other equilibrium points can be determined through implicit equations and its stability is difficult to determine, so it is investigated through simulation. Based on this, a simulation is presented and four conditions are obtained. Of the four conditions, there are two equilibrium points that experience a fold bifurcation (saddle-node).

REFERENCES

Cain, J. W., & Reynolds, A. M. (2010). *Ordinary and partial differential equations*. NC-ND, Virginia.

Chen, W. (2022). Stability and bifurcation analysis of a predator-prey model with michaelis-menten type prey harvesting. *Journal of Applied Mathematics and Physics*, *10*(02), 504–526. https://doi.org/10.4236/jamp.2022.102038

Doust, R. M. H., & Saraj, M. (2015). The logistic modeling population: Having harvesting factor. *Yugoslav Journal of Operations Research*, *25*(1), 107–115. https://doi.org/10.2298/YJOR130515038R

Etoua, R. M., & Rousseau, C. (2010). Bifurcation analysis of a generalized gause model with prey harvesting and a generalized holling response function of type III. *Journal of Differential Equations*, *249*(9), 2316–2356. https://doi.org/10.1016/j.jde.2010.06.021

Hale, J. K., & Kocak, H. (1991). *Dynamics and bifurcations*. Springer-Verlag New York, United States of America.

Hidayati, T. (2018). Kestabilan model populasi mangsa pemangsa. *Delta: Jurnal Ilmiah Pendidikan Matematika*, *6*(1), 38–46.

Karim, M. A., & Yulida, Y. (2019). Prediksi jumlah penduduk kalimantan selatan menggunakan metode nonlinear least-squares. *Media Bina Ilmiah*, *14*(5), 2605–2610.

Kuznetsov, Y. A. (1998). *Element of applied bifurcation theory*. Springer-Verlag, New York, USA.

Ludwig, D., Jones, D. D., & Holling, C. S. (1978). *Qualitative analysis of insect outbreak systems: The spruce budworm and forest*. *47*(1), 315–332.

Martcheva, M. (2015). *An introduction to mathematical epidemiology*. Springer, New York.

Maziun, N. A., & Subchan. (2020). Stability and bifurcation analysis of time delayed prey-predator system with holling type-III response function. *International Journal of Computing Science and Applied Mathematics*, *6*(2), 59–65.

Ndii, M. Z. (2018). *Pemodelan matematika dinamika populasi dan penyebaran penyakit*. Deepublish, Sleman.

Nurrobi, F., Yulida, Y., & Faisal. (2017). Bifurkasi pada model logistik dengan faktor pemanenan konstan. *Seminar Nasional Matematika Dan Terapannya I Program Studi Matematika FMIPA ULM Banjarbaru*, 36–41.

Olver, P. J., & Shikiban. (2003). *Appllied mathematics*. Springer Cambridge: University Press.

Otto, S., & Day, T. (2007). *A biologist's guide to mathematical modeling in ecology and evolution*. Princeton University Press, New Jersey.

Paul, S., Mondal, S. P., & Bhattacharya, P. (2016). Discussion on fuzzy quota harvesting model in fuzzy environment: Fuzzy differential equation approach. *Modeling Earth Systems and Environment*, *2*(2), 1–15. https://doi.org/10.1007/s40808-016-0113-y

Rahmi, E., & Panigoro, H. S. (2017). Pengaruh pemanenan terhadap model verhulst dengan efek allee. *SEMIRATA MIPAnet Vol. 1*, 105–112.

Robeva, R., & Murrugarra, D. (2016). The spruce budworm and forest : A qualitative comparison of ODE and Boolean models. *Letters in Biomathematics*, *3*(1), 1–12. https://doi.org/10.1080/23737867.2016.1197804

Tsoularis, A., Campus, A., & Zealand, N. (2001). Analysis of logistic growth models. *Res. Lett. Inf. Math. Sci*, *2*, 23–46.

Yulida, Y., & Karim, M. A. (2021). Prediction of rice consumption in south kalimantan by considering population growth rate. *IOP Conference Series: Earth and Environmental Science*, *758*(1). https://doi.org/10.1088/1755-1315/758/1/012022