Desimal: Jurnal Matematika Vol 4 No 2 (2021) 225-230

Contents lists available at DJM

DESIMAL: JURNAL MATEMATIKA

p-ISSN: 2613-9073 (print), e-ISSN: 2613-9081 (online), DOI 10.24042/djm

http://ejournal.radenintan.ac.id/index.php/desimal/index

Partition dimension of disjoint union of complete bipartite graphs

Debi Oktia Haryeni[1*], Edy Tri Baskoro[2], Suhadi Wido Saputro[2]

[1] Indonesia Defense University, Indonesia.
[2] Institut Teknologi Bandung, Indonesia.

ARTICLE INFO

Article History
Received : 20-05-2021
Revised : 02-06-2021
Accepted : 10-07-2021
Published : 31-07-2021

Keywords:
Resolvability; Metric Dimension; Partition Dimension; Disconnected Graph; Forest; Complete Bipartite Graph.

*Correspondence: E-mail:
debi.haryeni@idu.ac.id

Doi:
10.24042/djm.v4i2.10190

ABSTRACT

For any (not necessary connected) graph $G(V,E)$ and $A \subseteq V(G)$, the distance of a vertex $x \in V(G)$ and A is $d(x,A) = min\{d(x,a): a \in A\}$. A subset of vertices A resolves two vertices $x, y \in V(G)$ if $d(x,A) \neq d(y,A)$. For an ordered partition $\Lambda = \{A_1, A_2, \dots, A_k\}$ of $V(G)$, if all $d(x,A_i) < \infty$ for all $x \in V(G)$, then the representation of x under Λ is $r(x|\Lambda) = (d(x,A_1), d(x,A_2), \dots, d(x,A_k))$. Such a partition Λ is a resolving partition of G if every two distinct vertices $x, y \in V(G)$ are resolved by A_i for some $i \in [1,k]$. The smallest cardinality of a resolving partition Λ is called a partition dimension of G and denoted by $pd(G)$ or $pdd(G)$ for connected or disconnected G, respectively. If G have no resolving k −partition, then $pdd(G) = \infty$. In this paper, we studied the partition dimension of disjoint union of complete bipartite graph, namely $tK_{m,n}$ where $t \geq 1$ and $m \geq n \geq 2$. We gave the necessary condition such that the partition dimension of $tK_{m,n}$ are finite. Furthermore, we also derived the necessary and sufficient conditions such that $pdd(tK_{m,n})$ is either equal to m or m+1.

http://ejournal.radenintan.ac.id/index.php/desimal/index

INTRODUCTION

The idea of resolvability and location in graph were described independently by Slater in 1975 and Harary & Melter in 1976, to establish the same structure in a graph. This concept is then known as the metric dimension of a graph. In 1998 Chartrand *et al.* introduced the partition dimension parameter to possibly gain insight into metric dimension. Recently in 2015, Haryeni *et al.* generalized the definition of the partition dimension of a graph such that it can be also applied to disconnected graphs.

For any (not necessary connected) graph $G(V,E)$ and $A \subseteq V(G)$, the *distance* of a vertex $x \in V(G)$ and A, denoted by $d(x,A)$, is $\min\{d(x,a): a \in A\}$. If $d(x,A) \neq d(y,A)$, then we say A resolves two vertices $x, y \in V(G)$. For an ordered partition $\Lambda = \{A_1, A_2, \dots, A_k\}$ of $V(G)$, if all $d(x,A_i) < \infty$ for all $x \in V(G)$, then define the *representation* of x under Λ as $(d(x,A_1), d(x,A_2), \dots, d(x,A_k))$, and denoted by $r(x|\Lambda)$. A partition Λ is a r*esolving partition* of G if every two distinct vertices $x, y \in V(G)$ are resolved by some A_i, or in short $d(x, A_i) \neq d(y, A_i)$ for some $i \in [1, k]$. The smallest cardinality of a resolving partition Λ in G is called a *partition dimension* of G. We use the notation $pd(G)$ (or $pdd(G)$, respectively) for the partition dimension of connected (or disconnected) G. In case of G having no resolving k −partition, then define $pdd(G) = \infty$.

For connected graph, some authors have characterized the graphs of order n with certain partition dimension, namely for $pd(G) \in \{2, n-1, n\}$ by Chartrand et al. (2000), $pd(G) = n - 2$ by Tomescu (2008) and $pd(G) = n - 3$ by Baskoro & Haryeni (2020). The partition dimension of graphs with some graph operations also have been studied, such as for Cartesian product by Yero et a. (2010) and for strong product by Yero et al. (2014).

For the disconnected graphs G, many results in determining $pdd(G)$ have been obtained such as for linear forest mP_n, $\bigcup_{i=1}^{t} P_{n_i}$, and $K_3 \cup mP_n$ (Haryeni et al. 2017), and for disjoin union of star $\bigcup_{i=1}^{t} K_{1,n_i}$, double star forest $mT(r,s)$, and disjoint union of cycles (Haryeni et al., 2015). There were also some results on the partition dimension of two-component graphs (Haryeni et al., 2017). In this paper, we determine the partition dimension of a disjoint union of complete bipartite graph $tK_{m,n}$, where $m \geq n \geq 2$.

The following known results will be used in the main part of this paper.

Lemma 1. (Chartrand et al., 2000) Let Λ be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then u and v belong to distinct partition classes of Λ.

Furthermore, the two vertices u and v satisfying Lemma 1 are called *twin vertices*.

Theorem 1. (Chartrand et al., 2000) Let G be a connected bipartite graph with partite sets V_1 and V_2 of cardinalities m and n, respectively. Then

1) $pd(G) \leq m + 1$, if $m = n$ and
2) $pd(G) \leq \max\{m, n\}$, if $m \neq n$.

Moreover, equality holds in 1) or 2), if and only if G is complete bipartite graph.

Theorem 2. (Haryeni, Baskoro, 2017) Let $G = \bigcup_{i=1}^{m} G_i$. If $pdd(G) < \infty$, then we have that $\max\{pd(G_i) : 1 \leq i \leq m\} \leq pdd(G) \leq \min\{|V(G_i)| : 1 \leq i \leq m\}$.

Definition 1. (Haryeni, Baskoro, 2017) For $m \geq 1$, let $G = \bigcup_{i=1}^{m} G_i$ and $\Lambda = \{A_1, A_2, \ldots, A_k\}$ be a resolving partition of G. For any integer $t \geq 1$, a vertex $v \in V(G)$ is defined as a t −distance vertex if $d(v, A_j) = 0$ or t for any $A_j \in \Lambda$.

METHOD

In this paper, we determine the partition dimension of a disjoint union of complete bipartite graph $G = tK_{m,n}$, where $m \geq n \geq 2$. In general, to show that $pdd(G) = k$ for some k, we need to prove that the upper bound and the lower bound of the partition dimension of G is equal to k. To prove that $pdd(G) \leq k$, we define a partition of the vertices of G with k elements such that every vertex of G admits distinct representations with respect to such partition. To show that $pdd(G) \geq k$, we can prove by a contradiction. We assume that there exists a resolving $(k-1)$ −partition of G such that for any definition of such partition it always leads to the contradiction. By this procedure, we can conclude that $pdd(G) = k$.

RESULTS AND DISCUSSION

In this section we will determine the partition dimension of a disjoint union of $t \geq 1$ copies of complete bipartite graph $tK_{m,n}$, where $m \geq n \geq 2$. We begin this part with some related lemmas, as follows.

Lemma 2. Let $K_{m,n}$ be a complete bipartite graph where $m > n \geq 2$. In any minimum resolving partition of $K_{m,n}$, then there are at least n vertices which are $1-$distance vertices. Furthermore, there exists exactly one $1-$distance vertex in any resolving $(m+1)-$partition of $K_{m,n}$.

Proof. For $m > n \geq 2$, let the set of vertices and edges of $K_{m,n}$ be
$V(K_{m,n}) = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$,
$E(K_{m,n}) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

By Theorem 1, $pd(K_{m,n}) = m$. Let $\Lambda_1 = \{A_1, A_2, \dots, A_m\}$ be any minimum resolving partition of $K_{m,n}$. Since $d(u_i, x) = d(u_j, x)$ for any $x \in V(K_{m,n}) - \{u_i, u_j\}$, u_i and u_j belong to distinct partition class of Λ_1 for every $i, j \in [1, m]$, by Lemma 1. Without loss of generality, assume that $u_i \in A_i$ for any $i \in [1, m]$. By a similar reason, since $d(v_i, y) = d(v_j, y)$ for any $y \in V(K_{m,n}) - \{v_i, v_j\}$, we can assume that $v_j \in A_j$ for any $j \in [1, n]$. Therefore, v_j is $1-$distance vertex for all $j \in [1, n]$.

Now, let $\Lambda_2 = \{A_1, A_2, \dots, A_{m+1}\}$ be a resolving $(m+1)-$partition of $K_{m,n}$. Without loss of generality assume that $u_i \in A_i$ for any $i \in [1, m]$. Since $|\Lambda_2| = m+1$, there exists $v_j \in A_{m+1}$ for some $j \in [1, n]$. Thus, we may assume that $v_1 \in A_{m+1}$ and $v_j \in A_{j-1}$ for all $j \in [2, n]$. This implies that v_1 is the only $1-$distance vertex with respect to Λ_2.

Lemma 3. For integer $m \geq 2$, let $K_{m,m}$ be a complete bipartite graph. Then, in any minimum resolving partition of $K_{m,m}$ there are exactly two $1-$distance vertices.

Proof. For $m \geq 2$, let the set of vertices and edges of $K_{m,m}$ be
$$V(K_{m,m}) = \{u_i, v_j : 1 \leq i, j \leq m\},$$
$$E(K_{m,m}) = \{u_i v_j : 1 \leq i, j \leq m\}.$$

By using Theorem 1, $pd(K_{m,m}) = m+1$. Let $\Lambda = \{A_1, A_2, \dots, A_{m+1}\}$ be any minimum resolving partition of $K_{m,m}$. Note that u_i and u_j are twin vertices with respect to the partition Λ. Therefore, u_i and u_j belong to distinct partition class of Λ_1 for every $i, j \in [1, m]$ by using Lemma 1. Without loss of generality, assume that $u_i \in A_i$ for any $i \in [1, m]$. Since $|\Lambda| = m+1$, there exists $v_j \in V_2$ such that $v_j \in A_{m+1}$. Thus, we may assume that $v_1 \in A_{m+1}$. By a similar reason, since $d(v_i, y) = d(v_j, y)$ for any $y \in V(K_{m,m}) - \{v_i, v_j\}$, assume that $v_j \in A_{j-1}$ for all $j \in [2, m]$. Therefore, u_m and v_1 are $1-$distance vertices.

From now on, let $G = tK_{m,n}$ where $t \geq 1$ and $m \geq n \geq 2$. Let the set of vertices and edges of G be
$$V(G) = \{u_{i,j}, v_{i,k} : 1 \leq i \leq t, 1 \leq j \leq m, 1 \leq k \leq n\},$$
$$E(G) = \{u_{i,j} v_{i,k} : 1 \leq i \leq t, 1 \leq j \leq m, 1 \leq k \leq n\}.$$
and each component i^{th} of G has two partite sets V_{i1} and V_{i2} with cardinality m and n, respectively.

In the following result, we give the necessary condition for the graph $G = tK_{m,n}$ where $m \geq n \geq 2$ such that $pdd(G)$ is finite.

Theorem 3. For integer $t \geq 1$ and $m \geq n \geq 2$, if $pdd(tK_{m,n}) < \infty$, then $t \leq \frac{(m+n)!}{m!n!}$ for $m > n$, or $t \leq \frac{(2m)!}{2m!^2}$ for $m = n$.

Proof. let $G = tK_{m,n}$ with $t \geq 1$ and $m \geq n \geq 2$. If $pdd(tK_{m,n}) < \infty$, then to maximize the value of t, assume that $pdd(G) = m+n$. Now let $\Lambda = \{A_1, A_2, \dots, A_{m+n}\}$ be a resolving partition of G. Since $|\Lambda| = m+n = |V(G)|$, every vertex in each component of G is the singleton vertex in A_k for all $k \in [1, m+n]$. Thus for $m > n$, we have there are at most $\binom{m+n}{m}$ different ways to distribute the vertices of each component G into Λ. Moreover, for any two vertices $u, v \in A_p$ in which $u \in V_{1i}$ and $v \in V_{2j}$ there exists at least one $q \in [1, m+n] - \{p\}$ satisfying $d(u, A_q) = 1 \neq 2 = d(v, A_q)$.

Furthermore, for $m = n$, let $S_{ij} = \{s: x \in V_{ij}$ contained in $A_s\}$ where $1 \leq i \leq t$ and $1 \leq j \leq 2$. Thus, for $\binom{2m}{m}$ ways to distribute the vertices of each component of G into Λ, there exists $i \neq k$ such that $S_{i1} = S_{k2}$ and $S_{i2} = S_{k1}$. This implies that the representations of vertices in the i^{th} component are equal to the representations of vertices in k^{th} component of G. Therefore, $t \leq \frac{(2m)!}{2m!^2}$ for $m = n$.

In the next results, we give the characterization of the graph $G = tK_{m,n}$ with $m \geq n \geq 2$ such that the partition dimension G is either m or $m + 1$.

Theorem 3. For integer $t \geq 1$ and $m \geq n \geq 2$, then

$$pdd(G) = \begin{cases} m, & if\ and\ only\ if\ m > n \\ & and\ t \leq \left\lfloor \frac{m}{n} \right\rfloor, \\ m+1, & if\ and\ only\ if\ (m > n\ and \\ & \left\lfloor \frac{m}{n} \right\rfloor + 1 \leq t \leq m+1)\ or \\ & \left(m = n\ and\ \ t \leq \left\lfloor \frac{m+1}{2} \right\rfloor\right). \end{cases}$$

Proof. For $t \geq 1$, let $G = tK_{m,n}$ where $m \geq n \geq 2$. Note that $pdd(G) \geq m$ by Theorems 1 and 2. Now, we distinguish the following cases.

Case 1. $m > n$ and $t \leq \left\lfloor \frac{m}{n} \right\rfloor$. We will show that $pdd(G) \leq m$. Define a partition $\Lambda_1 = \{A_1, A_2, \ldots, A_m\}$ of G induced by the function $f: V(G) \to \{1, 2, \ldots, m\}$ as follows.

$$f(u_{i,j}) = j, \text{ for any } i \in [1,t], j \in [1,m],$$

$$f(v_{i,k}) = (i-1)n + k, \quad \text{for any} \quad i \in [1,t], k \in [1,n].$$

Note that $f(x) = i$ means that $x \in A_i$. Let us show that Λ_1 is a resolving partition of G. We consider any two distinct vertices $x, y \in V(G)$ in A_p for some $p \in [1, m]$. If $x = u_{i,j}$ and $y = u_{k,j}$ for a distinct $i, k \in [1,t]$ and $j \in [1,m]$, then $d(x, A_q) = 1 \neq 2 = d(y, A_q)$ for some $q \in [(i-1)n + 1, in]$. If $x = u_{i,j}$ and $y = v_{k,l}$ for some $i, k \in [1,t], j \in [1,m]$, and $l \in [1,n]$, then $j = (k-1)n + l$ and $d(x, A_q) = 2 \neq 1 = d(y, A_q)$ for some $q \in ([1,m] - [(i-1)n+1, in])$. Therefore, $r(x|\Lambda_1) \neq r(y|\Lambda_1)$ for any two vertices $x, y \in V(G)$ and so that Λ_1 is a resolving partition of G.

Now we will prove the reverse direction. For $m \geq n \geq 2$ and $t \geq 1$, let $G = tK_{m,n}$ and $pdd(G) = m$. By Theorem 1, then we obtain that $m > n$. Furthermore, any component of G has at least n vertices as $1-$distance vertex for any $m-$resolving partition of G, by Lemma 1. This implies that $t \leq \left\lfloor \frac{m}{n} \right\rfloor$.

Case 2. $m > n$ and $\left\lfloor \frac{m}{n} \right\rfloor + 1 \leq t \leq m + 1$. We will show that $pdd(G) = m + 1$. By considering Case 1, then $pdd(G) \geq m + 1$. To show the upper bound, let $\Lambda_2 = \{A_1, A_2, \ldots, A_{m+1}\}$ be a partition of G induced by the function $g: V(G) \to \{1, 2, \ldots, m+1\}$, as follows.

$$g(x) = \begin{cases} i + j \bmod (m+1), & if\ x = u_{i,j}, \\ i + k - 1 \bmod (m+1), & if\ x = v_{i,k}, \end{cases}$$

where $i \in [1,t], j \in [1,m]$, and $k \in [1,n]$. Note that $g(x) = 0$ means that $x \in A_{m+1}$ and $g(x) = i$ means that $x \in A_i$ for some $i \in [1,m]$. Consider any two distinct vertices $x, y \in V(G)$ in A_p for some $p \in [1, m+1]$. If $x = u_{i,j}$ and $y = u_{k,l}$ where

$i \neq k$ and $j \neq l$, then $d(x, A_q) = 1 \neq 2 = d(y, A_q)$ for some $q \in [1, m+1] - [i \bmod (m+1), i+n-1 \bmod (m+1)]$. If $x = u_{i,j}$ and $y = v_{k,l}$, then $d(x, A_q) = 1 \neq 2 = d(y, A_q)$ for some $q \in [i+1, i+m \bmod (m+1)]$). Hence for any two vertices $x, y \in V(G)$, we obtain that $r(x|\Lambda_2) \neq r(y|\Lambda_2)$ and so that Λ_2 is a resolving partition of G.

To show the reverse direction, let $pdd(tK_{m,n}) = m$, where $m > n \geq 2$ and $t \geq 1$. By considering Case 1, then $t \geq \left\lfloor \frac{m}{n} \right\rfloor + 1$. Furthermore, any component of $tK_{m,n}$ has exactly one $1-$distance vertex for any $(m+1)-$resolving partition of G, by Lemma 1. Thus, we can conclude that $t \leq m+1$.

Case 3. $m = n$ and $t \leq \left\lfloor \frac{m+1}{2} \right\rfloor$. By Theorems 1 and 2, then $pdd(G) \geq m+1$. Let $\Lambda_3 = \{A_1, A_2, \dots, A_{m+1}\}$ be a partition of G induced by the function $h: V(G) \to \{1, 2, \dots, m+1\}$ as follows.

$$h(x) = \begin{cases} 2i-1, & if\ x = u_{i,1}, \\ 2i, & if\ x = v_{i,1}, \\ 2i+j-1 \bmod (m+1), & if\ x \in \{u_{i,j}, v_{i,j}\}, \end{cases}$$

where $i \in [1, t]$ and $j \in [2, m]$. Note that $h(x) = 0$ means that $x \in A_{m+1}$ and $h(x) = i$ means that $x \in A_i$ for some $i \in [1, m]$. We consider any two distinct vertices $x, y \in V(G)$ in A_p for some $p \in [1, m+1]$. If $x \in \{u_{i,1}, v_{i,1}\}$ and $y \in \{u_{j,k}, v_{j,k}\}$ for some $k \in [2, m]$, then $d(x, A_q) = 1 \neq 2 = d(y, A_q)$ for some $q \in \{2i-1, 2i\}$. If $(x = u_{i,j}$ and $y = u_{a,b})$, *or* $(x = v_{i,j}$ and $y = v_{a,b})$ for some $j, b \in [2, m]$, then $d(x, A_q) = 2 \neq 1 = d(y, A_q)$ for $q = 2i-1$ or $q = 2i$, respectively. If $x = u_{i,j}$ and $y = v_{a,b}$ for some $j, b \in [2, m]$, then $d(x, A_{2i-1}) = 2 \neq 1 = d(y, A_{2i-1})$. This implies that $r(x|\Lambda_3) \neq r(y|\Lambda_3)$ for any two vertices $x, y \in V(G)$ and so that Λ_3 is a resolving $(m+1)-$partition of G.

To show the reverse direction, let $pdd(tK_{m,n}) = m+1$, where $m, n \geq 2$ and $t \geq 1$. By considering Case 2, then $m = n$. Note that for each component of $tK_{m,n}$, there are exactly two $1-$distance vertices for any resolving $(m+1)-$partition of G by Lemma 2. Therefore, we can conclude that $t \leq \left\lfloor \frac{m+1}{2} \right\rfloor$.

CONCLUSIONS AND SUGGESTIONS

Based on the results above, we can conclude that if the partition dimension of $tK_{m,n}$ is finite, then $t \leq \frac{(m+n)!}{m!n!}$ for $m > n \geq 2$, or $t \leq \frac{(2m)!}{2m!^2}$ for $m = n \geq 2$. Furthermore, we conclude from the result in Theorem 3 that the partition dimension of $tK_{m,n}$ is equal to m if and only if $m > n \geq 2$ and $t \leq \left\lfloor \frac{m}{n} \right\rfloor$, and the partition dimension of $tK_{m,n}$ is equal to $m+1$ if and only if $(m > n \geq 2$ and $\left\lfloor \frac{m}{n} \right\rfloor + 1 \leq t \leq m+1)$ or $(m = n$ and $t \leq \left\lfloor \frac{m+1}{2} \right\rfloor)$.

There are some open problems related to this topic, namely to determine the partition dimension of disjoint union of complete bipartite graph $tK_{m,n}$ for other t defined in Theorem 3. Moreover, we can also study the partition dimension of disjoint union of complete bipartite graph with different order, namely $G = \bigcup_{i=1}^{t} K_{m,n_i}$ where $n_i \neq n_j$ for some $i, j \in [1, t]$.

REFERENCES

Baskoro, E. T. & Haryeni, D. O. (2020). All graphs of order $n \geq 11$ and diameter 2 with partition dimension $n-3$. *Heliyon,* 6 e03694.

Chartrand, G. & Zhang, P. (2003). The theory and application of resolvability in graphs. *Congr. Numer.*, 160, 47-68.

Chartrand, G., Salehi, E., & Zhang, P. (2000). The partition dimension of a graph. *Aequationes Math*., 59, 45-54.

Harary, F. & Melter, R. A. (1976). On the metric dimension of graph. *Ars. Combin*., 2, 191-195.

Haryeni, D. O. & Baskoro, E. T. (2015). Partition dimension of some classes of homogeneous disconnected graphs. *Procedia Compute. Sci*., 74, 73-78.

Haryeni, D. O., Baskoro, E. T., Saputro, S. W., Baca, M., & Semanicova-Fenovcikova, A. (2017). On the partition dimension of two-component graphs. *Proc. Indian Acad. Sci. (Math. Sci.)*, 127 (5), 755-767.

Haryeni, D. O., Baskoro, E. T., & Saputro, S. W. (2017). On the partition dimension of disconnected graphs. *J. Math. Fund. Sci*., 49 (1), 18-32.

Slater, P. J. (1975). Leaves of trees. *Congr. Numer*., 14, 549-559.

Tomescu, I. (2008). Discrepancies between metric dimension and partition dimension of a connected graph. *Discrete Math*., 308, 5026-5031.

Yero, I. G., Kuziak, D. & Rodriguez-Velazquez, J. A. (2010). A note on the partition dimension of Cartesian product graphs. *Appl. Math. Comput*., 217, 3571-3574.

Yero, I. G., Jakovac, M., Kuziak D., & Taranenko, A. (2014). The partition dimension of strong product graphs and Cartesian product graphs. *Discrete Math*., 331, 43-52.