The relationship of the formulas for the number of connected vertices labeled graphs with order five and order six without loops

Amanto*, Notiragayu, La Zakaria, Wamiliana

Universitas Lampung, Indonesia

ARTICLE INFO

*Correspondence: E-mail:
pakamanto@gmail.com

ABSTRACT

Given a graph with n points and m lines. If each vertex is labeled, then it can be constructed many graphs, connected, or disconnected graphs. A graph G is called a connected graph if there is at least one path that connects a pair of vertices in G. In addition, the graph formed may be simple or not simple. A simple graph is a graph that does not contain loops or parallel lines. A loop is a line that connects a point to itself, and a parallel line is two or more lines that connect the same pair of points. This paper will discuss the relationship between the formula patterns for calculating the number of connected graphs labeled with vertices of order five and six without loops.

http://ejournal.radenintan.ac.id/index.php/desimal/index

INTRODUCTION

Many real-life problems can be represented using graph theory. Graph theory is one of the fields in mathematics that is widely used as a tool to represent problems. Many branches of science that use the concept of graph theory, including chemistry, biology, computer science, economics, engineering, and others. In the field of chemistry, for example, Burch (2018) used graphs to model chemical compounds to provide an overview of the physical properties of these chemical compounds; in the field of biology Mathur & Adlakha (2016) used a combination of trees to represent DNA, Huson & Bryant (2006) and Brandes & Cornelsen (2009) used a labeled tree to represent the evolution of taxa called phylogenetic trees; in the field of computer science Hsu & Lin (2008) comprehensively discussed graph concepts related to network design, and Etaiwi (2014) used the minimum spanning tree concept, complete graph, and cycle graph to generate a ciphertext and others.

A Graph G is defined as an unordered pair of $(V(G), E(G))$ where $(G) = \{v_1, v_2, ..., v_n\}$ is a set of points/vertices, (G), and $(G) = \{e_1, e_2, ..., e_n\}$ is the set of edges or lines

of G. An edge or line with the same starting point and ending point is called a loop, while a parallel line is two or more lines connecting the same points. A simple graph is a graph that does not contain loops or parallel lines, otherwise, the graph is called not simple. Vertices on a graph are generally used to represent cities/stations/airports, and so on, while lines are used to represent roads/railroads/airlines, and so on. For example, in a transportation problem, cities can be represented by vertices while roads connecting cities can be represented by lines. In addition, each line can be assigned a value that can represent non-structural information such as distance, cost, time, and others. Thus, by representing the problem in the form of a graph, the problems encountered can be easily visualized.

Graph enumeration was pioneered by Cayley (1874) when calculating the number of $C_n H_{2n+2}$ hydrocarbon isomers which turned out to be related to counting rooted trees in graphs. Harary & Palmer (1973) provided a technique for calculating the number of graphs in general. Wamiliana et al. (2017) conducted research to determine the number of disconnected vertices labeled graphs of order five without parallel lines. Amanto et al. (2017) provided a formula to determine the number of disconnected vertices labeled graphs with a maximum order of four. Furthermore, Amanto et al. (2018) provided a formula to calculate disconnected vertex labeled graph of order five with a maximum of six 3-parallel edges without loops. In the same year, Dracjat et al. (2018) conducted research to calculate the number of connected graphs labeled with vertices of order five with a maximum of two parallel lines or loops and a maximum of six non-parallel lines. Wamiliana et al. (2019) investigated a formula to calculate the number of vertices labeled graphs of order five with a maximum of five parallel lines

and no loops. Puri et al. (2021) provided a formula to calculate the number of connected graphs labeled with vertices of order 6 with a maximum of 30 lines without loops, and Putri et al. (2021) conducted research to determine the number of disconnected vertices labeled graphs of order 6 with a maximum of 20 parallel lines and no loops.

In this research, the patterns of the formula for the number of connected vertex labels of order five and six without loops will be observed to find the relationship between the formula.

METHOD

Given a graph with n vertices and m lines with each vertex is labeled. Then, many graphs can be constructed, whether connected or disconnected, simple or not simple. In the construction process, isomorphic graphs are only counted as one graph. Two graphs G and G' are said to be equivalent (or sometimes are called isomorphic) if there is a one-one correspondence between the vertices of the two graphs and between their lines so that if the line e is adjacent to the points u and v in G then the line e' in is G' also adjacent to the points u' and v' in G'.

Notation:
n = the number of vertices
m = the number of edges
g = the number of non-parallel edges
$g = m - \sum_{i=1} p_i$
p_i = the number of i parallel edges; $i \geq 2$
j_i = coefficient for the number of i – parallel edges
t = the number of edges that connect the same pairs of vertices (parallel edges are counted as one).
$m = \sum_{i=2} j_i \cdot p_i + g$
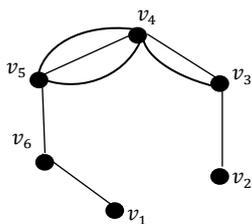
Figure 1. An Example of a Graph of Order Six with $m = 8$, $g = 3$, $p_2=1$, $p_3=1$, $j_2=1$, and $j_3=1$.

Based on Figure 1, the total numbers of edges $m = 8$, the number of non-parallel edges g is 3, the number of two parallel edges p_2 is 1 ($j_2=1$), and the number of three parallel edges p_3 is also 1 ($j_3=1$).

$$m = j_2p_2 + j_3p_3 + g$$
$$= (1)(2) + (1)(3) + 3$$
$$= 8$$

To find the formula for the number of connected vertices labeled graph of order six without loops, first, constructed the patterns of the graph by comparing it with the pattern of order five and then find which patterns do not exist in order five. Next, calculate the number of graphs obtained in every pattern and then predict and determine the formula by comparing it with the formula of order five.

Table 1. Comparison of Some Patterns of Connected Vertex Labeled Graphs of Order Five and Six without Loops

t	$n = 5$	$n = 6$
$t=n-1$ $m=n-1$		
$t=n-1$ $m=n$		

t	$n = 5$	$n = 6$
$t = n$ $m = n$		
$t = n$ $m=n+1$		
$t=n+1$ $m=n+1$		
$t=n+1$ $m=n+2$		
$t=n+2$ $m=n+2$		
$t=n+2$ $m=n+3$		

t=n+3
m=n+3

t=n+3
m=n+4

t=n+4
m=n+4

t=n+4
m=n+5
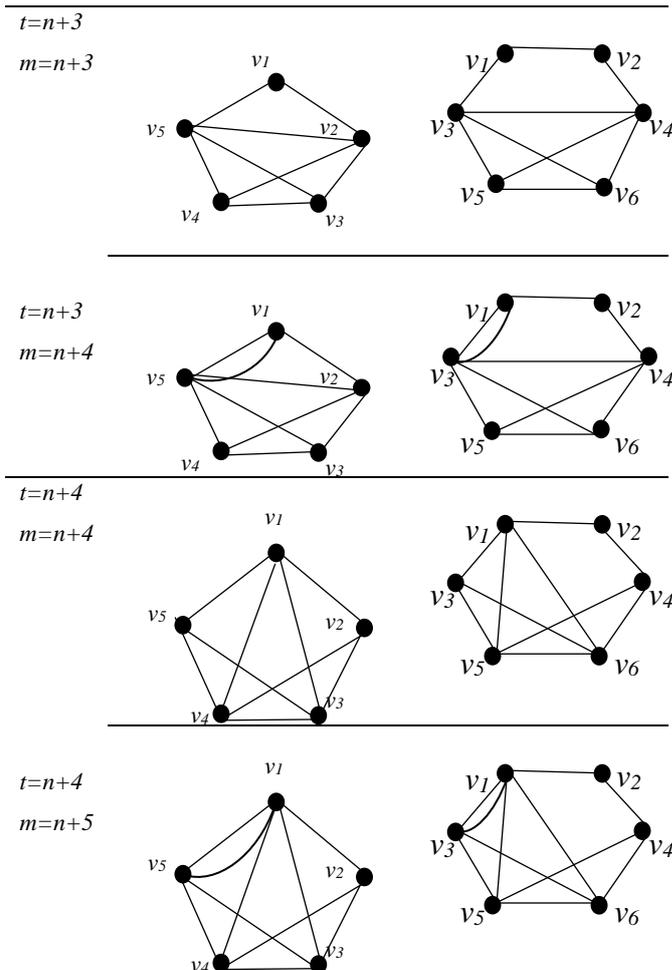
Table 1 only provides some patterns due to many patterns obtained and space limitations. For example, for order five (n=5), for t = n+4 and m = n+6, some patterns obtained are as follow:
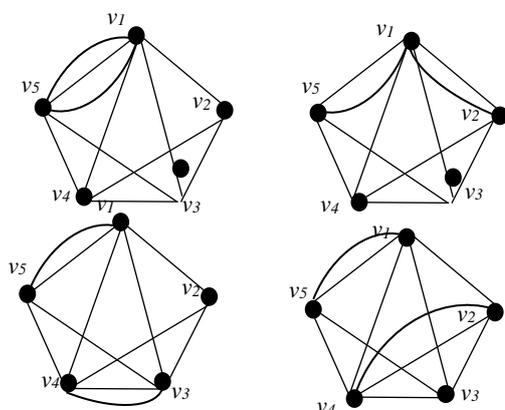
Figure 2. Some Patterns of Connected Vertex Labeled Graphs of Order Six Without Loops for t=n+4 and m=n+6.

RESULTS AND DISCUSSION

From the observation, it is found that after grouping the graphs obtained by using t and m, the number of graphs is described in Table 2.

Table 2. Comparison of the Number of Connected Vertices Labeled Graphs of Order Five and Six without Loops.

m	t			
	5		**6**	
	n=5	**n=6**	**n=5**	**n=6**
5	1x222	1 x 1296		
6	5x222	5 x 1296	1x205	1 x 1980
7	15x222	15 x 1296	6x205	6 x 1980
8	35x222	35 x 1296	21x205	21 x 1980
9	70x222	70 x 1296	56x205	56 x 1980
10	126x222	126 x 1296	126x205	126 x 1980
11		210 x 1296		252 x 1980
12		330 x 1296		462 x 1980
13		495 x 1296		792 x 1980
14		715 x 1296		1287 x 1980
15		1001 x 1296		2002 x 1980

m	t			
	7		**8**	
	n=5	**n=6**	**n=5**	**n=6**
7	1x110	1 x 3330		
8	7x110	7 x 3330	1x45	1 x 4620
9	28x110	28 x 3330	8x45	8 x 4620
10	84x110	84 x 3330	36x45	36 x 4620
11		210 x 3330		120 x 4620
12		462 x 3330		330 x 4620
13		924 x 3330		792 x 4620
14		1716 x 3330		1716 x 4620
15		3003 x 3330		3432 x 4620

m	t			
	9		**10**	
	n=5	**n=6**	**n=5**	**n=6**
9	1x10	1 x 6660		
10	9x10	9 x 6660	1x1	1 x 2460
11		45 x 6660		10 x 2460
12		165 x 6660		55 x 2460
13		495 x 6660		220 x 2460
14		1287 x 6660		715 x 2460
15		3003 x 6660		2002 x 2460

The authors provide the results for the number of connected vertices labeled

graph of order five and six. For order 5, we only observed for $4 \leq m \leq 10$, and $5 \leq m \leq 15$ for order six. However, Table 2 is not including $m=4$ for order five because it cannot compare it to order six if $m=4$ (if $m=4$ and $n=6$, then the graph will be disconnected).

Let $N(G_{n,m,t})$ be the number of connected vertices labeled graph of order n ($n=5, 6$) without loops, with m edges and t denotes the number of edges that connect the same pair of vertices.

Based on Table 2, it can be seen that in every column there is a sequence of numbers that multiply by a fixed number, for example for $n=5$ and $t=5$ it is obtained sequence 1, 5, 15, 35, 70, 126, and this sequence is multiplied by 222.

```
1      5      15      35      70      126
   4       10      20      35      56
      6       10      15      21
         4       5      6
            1      1
```

Theorem 1:

Given $n=5$, $4 \leq m \leq 10$, and $t=5$, the number of vertices labeled connected graphs of order 5 without loop is $N(G_{n,m,t}) = 222 \times C_4^{(m-1)}$.

Proof:

See the sequence of numbers above. The fixed difference occurs in the fourth level, therefore the polynomial that is related to that sequence is a polynomial of order four

$$P_4(m) = \alpha_4 m^4 + \alpha_3 m^3 + \alpha_2 m^2 + \alpha_1 m + \alpha_0.$$

Thus, we get the following system of the equation:

$$222 = 625\alpha_4 + 125\alpha_3 + 25\alpha_2 + 5\alpha_1 + \alpha_0 \quad (1)$$
$$1110 = 1296\alpha_4 + 216\alpha_3 + 36\alpha_2 + 6\alpha_1 + \alpha_0 \quad (2)$$
$$3330 = 2401\alpha_4 + 343\alpha_3 + 49\alpha_2 + 7\alpha_1 + \alpha_0 \quad (3)$$
$$7770 = 4096\alpha_4 + 512\alpha_3 + 64\alpha_2 + 8\alpha_1 + \alpha_0 \quad (4)$$
$$15540 = 6561\alpha_4 + 729\alpha_3 + 81\alpha_2 + 9\alpha_1 + \alpha_0 \quad (5)$$

By solving that system of **equation,** obtained:

$$P_4(m) = \frac{222}{24}m^4 - \frac{2220}{24}m^3 + \frac{7770}{24}m^2 - \frac{11100}{24}m + \frac{5328}{24}.$$

$$= \frac{222}{24}(m^4 - 10m^3 + 35m^2 - 50m + 24)$$

$$= \frac{222}{24}(m-1)(m-2)(m-3)(m-4)$$

$$= 222 \times C_4^{(m-1)}.$$

Table 3 shows the results for $n=5$ and $n=6$.

From Table 3, it can be seen that the number of graphs obtained make a pattern as: $N(G_{n,m,t}) = k_t \, C_{t-1}^{(m-1)}$, where for $n = 5$, $k_5=222$, $k_6= 205$, $k_7 = 110$, $k_8 = 45$, $k_9 = 10$, and $k_{10}=1$. For $n=6$, $k_5=1296$, $k_6= 1980$, $k_7 = 3330$, $k_8 = 4620$, $k_9 = 6660$, $k_{10} = 2640$, $k_{11} = 1155$, $k_{12} = 420$, $k_{13} = 150$, $k_{14} = 15$, $k_{15} = 1$. Note that for $n = 5$, maximal number of t is 10, and maximal number of t for $n= 6$ is 15. Thus, the formula for the number of connected vertices labeled graph of order five and six without loops differs only on the coefficient of t for every t, and when t is maximum then the formula is $C_{t-1}^{(m-1)}$.

Table 3. The Number of Connected Vertices Labeled Graphs of Order Five and Six without Parallel Edges.

t	n	
	5	6
5	$N(G_{5,m,5}) = 222 \times C_4^{(m-1)}$	$N(G_{6,m,5}) = 1296 \times C_4^{(m-1)}$
6	$N(G_{5,m,6}) = 205 \times C_5^{(m-1)}$	$N(G_{6,m,6}) = 1980 \times C_5^{(m-1)}$
7	$N(G_{5,m,7}) = 110 \times C_6^{(m-1)}$	$N(G_{6,m,7}) = 3330 \times C_6^{(m-1)}$
8	$N(G_{5,m,8}) = 45 \times C_7^{(m-1)}$	$N(G_{6,m,8}) = 4620 \times C_7^{(m-1)}$
9	$N(G_{5,m,9}) = 10 \times C_8^{(m-1)}$	$N(G_{6,m,9}) = 6660 \times C_8^{(m-1)}$
10	$N(G_{5,m,10}) = 1 \times C_9^{(m-1)}$	$N(G_{6,m,10}) = 2640 \times C_9^{(m-1)}$
11		$N(G_{6,m,11}) = 1155 \times C_{10}^{(m-1)}$
12		$N(G_{6,m,12}) = 420 \times C_{11}^{(m-1)}$
13		$N(G_{6,m,13}) = 150 \times C_{12}^{(m-1)}$
14		$N(G_{6,m,14}) = 15 \times C_{13}^{(m-1)}$
15		$N(G_{6,m,13}) = 1 \times C_{14}^{(m-1)}$

CONCLUSIONS AND SUGGESTIONS

Based on the discussion above, it can be concluded that the formula for the number of connected vertices labeled graphs of order five and six is $N(G_{n,m,t}) = k_t\ C_{t-1}^{(m-1)}$, where for $n = 5$, $k_5 = 222$, $k_6 = 205$, $k_7 = 110$, $k_8 = 45$, $k_9 = 10$, and $k_{10} = 1$. For $n = 6$, $k_5 = 1296$, $k_6 = 1980$, $k_7 = 3330$, $k_8 = 4620$, $k_9 = 6660$, $k_{10} = 2640$, $k_{11} = 1155$, $k_{12} = 420$, $k_{13} = 150$, $k_{14} = 15$, $k_{15} = 1$. Moreover, **it is** conclude**d** that if t is maximum, then for every order (five and six), $N(G_{n,m,t}) = C_{t-1}^{(m-1)}$.

ACKNOWLEDGEMENT

REFERENCES

Amanto, A., Wamiliana, W., & Efendi, M. F. N. (2018). The number of disconnected vertex labelled graphs of order five with maximum 3-paralel edges is six and contains no loops. *Konferensi Nasional Matematika 2018- Unibraw Malang*.

Amanto, A., Wamiliana, W., Usman, M., & Permatasari, R. (2017). Counting the number of disconnected vertex labelled graphs with order maximal four. *Sci.Int*, *29*(6), 1181–1186.

Brandes, U., & Cornelsen, S. (2009). Phylogenetic graph models beyond trees. *Discrete Applied Mathematics*, *157*(10). https://doi.org/10.1016/j.dam.2008.06.031

Burch, K. J. (2018). Chemical applications of graph theory. In *Mathematical Physics in Theoretical Chemistry*. https://doi.org/10.1016/B978-0-12-813651-5.00008-5

Cayley. (1874). On the mathematical theory of isomers. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *47*(314). https://doi.org/10.1080/14786447408641058

Dracjat, I., Wamiliana, W., Asmiati, A., & Amanto, A. (2018). Banyaknya graf terhubung berlabel titik berorde lima dengan garis paralel atau loop

maksimal dua serta garis non paralel maksimal enam. *Seminar Nasional Metode Kuantitatif II 2018*.

Etaiwi, W. M. Al. (2014). Encryption algorithm using graph theory. *Journal of Scientific Research and Reports*, *3*(19). https://doi.org/10.9734/JSRR/2014/11804

Harary, F., & Palmer, E. M. (1973). *Graphical enumeration*.

Hsu, L.-H., & Lin, C.-K. (2008). Graph theory and interconnection networks. In *Graph Theory and Interconnection Networks*. https://doi.org/10.1201/9781420044829

Huson, D. H., & Bryant, D. (2006). Application of phylogenetic networks in evolutionary studies. *Molecular Biology and Evolution*, *23*(2).

Mathur, R., & Adlakha, N. (2016). A graph theoretic model for prediction of reticulation events and phylogenetic networks for DNA sequences. *Egyptian Journal of Basic and Applied Sciences*, *3*(3). https://doi.org/10.1016/j.ejbas.2016.07.004

Puri, F. C., Wamiliana, Usman, M., Amanto, Ansori, M., & Antoni, Y. (2021). The formula to count the number of vertices labeled order six connected graphs with maximum thirty edges without loops. *Journal of Physics: Conference Series*, *1751*(1). https://doi.org/10.1088/1742-6596/1751/1/012023

Putri, D., Wamiliana, Fitriani, Faisol, A., & Dewi, K. S. (2021). Determining the number of disconnected vertices labeled graphs of order six with the maximum number twenty parallel edges and containing no loops. *Journal of Physics: Conference Series*, *1751*(1). https://doi.org/10.1088/1742-6596/1751/1/012024

Wamiliana, Nuryaman, A., Amanto,

Sutrisno, A., & Prayoga, N. A. (2019). Determining the number of connected vertices labelled graph of order five with maximum number of parallel edges is five and containing no loops. *Journal of Physics: Conference Series*, *1338*(1). https://doi.org/10.1088/1742-6596/1338/1/012043

Wamiliana, W., Amanto, A., & Tumpi Nagari, G. (2017). Counting the number of disconnected labeled graphs of order five without parallel edges. *INSIST*, *1*(1). https://doi.org/10.23960/ins.v1i1.7