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Algebraic and visual representation in solving mathematics problems based on empirical thinking

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Abstract

This study aims to investigate and analyze students' approaches to solving mathematics problems that are presented either visually or algebraically, without any prior guidance or hints. The goal is to capture and understand students' empirical verification thinking through their problem-solving trajectories. Students were encouraged to respond to these problems based on their own reasoning processes, observed through the steps they took in reaching their solutions. The research was conducted over two semesters in the Mathematics Education Department at Tanjungpura University, West Borneo, Indonesia. The problem sets were derived from the Researcher's Repertoire, national Teacher Professional Education test items, and the Flanders Mathematics Olympiad. Prior to administration, the problems underwent construct validity testing using Cramer's V test, yielding a coefficient value of 0.83, based on input from selected participants. The analysis focused on students' patterns of empirical verification thinking, the types of representations used, and the logical completeness of their solution steps. The findings indicate that students predominantly followed linear or meta-patterns in their reasoning, while their descriptive explanations exhibited diverse, non-linear approaches. Furthermore, the logical steps taken were often not clearly identifiable within standard forms of reasoning. In general, a higher use of visual representation corresponded with a reduction in varied representational thinking. Most responses were grounded in algebraic reasoning, with minimal visual representation and without relying on manipulation or hints. The visual elements included were used solely as components of the problem-solving process, rather than as tools for manipulation.

Keywords: No Hint Problem, Visual, Algebraic, Empirical Verification Thinking.

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Introduction

It is quite possible for people to speak fluently about a scene as they describe it, either by seeing it directly or in imagination. However, we propose that such fluent verbal descriptions are related only to what we term *surface features* of perception or images. Surface features are those aspects of an object or scene that require very little conscious effort to grasp. Deep features of perceptual objects, scenes, or of mental images are, to the contrary, those features that take considerable conscious effort to assimilate. The representational thinking of mathematics problem comprises of algebraic and visual. That is an analytic approach or a geometric point of view. The algebraic performance looks at symbolic manipulation, while the geometric appears concepts of in-depth understanding of the visual situation. From previous research (Rif'at, 1998) founded that geometric objects considered not inherently in-depth means as static features. In this research, the consideration is to develop in-depth visual and algebraic representations, independent of students look at the mathematics problems.

The research is to examine features of an object to describe the representation analytically or visually. The algebraic considered a symbolic expression based on logical relationship connected to the thought. According to Rif'at (2001), the visual representation is a model of thinking, understanding problem, and to articulate the examination of perception or imagination. Articulateness in the representational thinking and the steps of the solution tends to be a temporary variable and not linked to the clarity of thought. For example, the researcher gives a problem to the students to carry out their thinking but arriving at the algebraic-analytic solution and, at the same time becoming dull, and disturbing their thinking. There is hesitancy in demonstrating an explanation or exploration from the visual, while the algebraic thinking bounded in constructing communicative relationships in the solutions. It means that the student's visual thinking is holding and looking more accessible or recognizable geometry objects. That positioned the objects not for solving but describing the visual representation 'as enough as' possible in their images. They give some examples of particular cases recognizable to describe the object's features but can't see the whole situation.

An effort toward preparing the students' thinking requires a rethinking of algebraic and visual representation (Rif'at, 2018). The importance is to closing the mathematics opportunity gap and increasing the solutions' performance from the representations. That is to separate as much as of the two representations when solving the problems. For example, the students use each of the representation to solve any problems that presented visually or need to visualize them in making or arranging the solutions (Rif'at, Rachmat, Sugiatno, & Dede, 2019). In the algebraic or analytic representation, the students related two or more relationships in constructing either the visual or the algebraic but not simultaneously in both.

In a review of research on preparing the students' thinking noted that while many pre-service students expect to work in mathematics knowledge, the most have little knowledge or experience in the visual (Rif'at, 2019). That is, mental imagery needed a model of thinking related to the mathematics representations. But, no priming effect suggested mental images when solving the problems. The researcher observes that a reference to perceptual experience can distinguish the mental imagery from arithmetic (or algebra thinking). For example, the latest research discovered that the students draw a visual even though no need, and there is no visual activity anymore from problem represented visually.

The research has examined how to help the students develop thinking models in visual and algebraic representation, respectively, in solving the problems. For instance, that is to build up the mental imageries. That considered different aspects from a theoretical framework where the notion of a solution provides insights, provides a comprehensive understanding of the representations. According to Landa (1976), the notion is a unifying and a connective concept of flexibility thinking. Concerning to Landa (1976, p. 7), this research underlies thinking and performance in solving the problems using the representations, a mental operations viewed as a kind of imagery thinking "algebraic" and "visuals."

There are classes of mathematics problems for which necessary to execute the solutions in a well-structured. That is to re-formulate algebraic and geometric representations mentally to solve a mathematics problem, as cognitive activities, analyzed into operations of the algebra, and semi-analytic or visual. The theory of learning specifies taught not only knowledge but the thinking of representation as well. That is how to discover solutions and to think on their own. The emphasis is on cognitive operations of the representations which make up models of thinking, particularly by empirical verification.

With respect to the representations, proposed a number of solving strategies based on the models of thinking. That is to recognize the visual and algebraic thinking classified in many steps of different situations. The thinking models verified by mapping competencies as depicted in Table 1.

Table 1. A Mapping of Algebraic and Visual Representations Related to the Thinking Models

Mapping Competencies Based On Representation		
Algebraic representation	Visual representation	Thinking model
Formulating	Formulating	Transforming
Connecting	Relating	Simplifying
Modeling	Modifying	Manipulating

The students thinking models in the first reaction of a mathematics problem contain visual responses to transform the representations in the solving. But, they used algebra knowledge, an algorithmic or analytic manipulation point of view. The second reaction is to simplify critical attributes of the geometric shapes, i.e., the students connect the visual situation to the algebraic concepts and relating the situation analytically to solve the problems. In mathematics, the critical attributes stem from the definition of the concept (Tsamir et al., 2008) that looks merely memorizing. For example, the students asked to explain the visual relationship, as described in Figure 1. The students observed the representation but using algebra formula to relate a possible one, i.e., the area of the triangles and the hexagon.

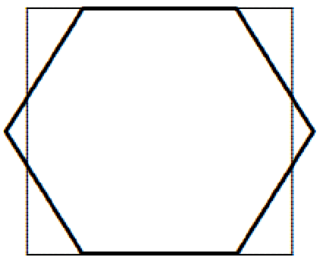


Figure 1. A Square of Side One Intersects with a Hexagon

When asked to determine the area of the intersections, the students and some lecturers can't answer the problem because their algebra knowledge mainly influences the solution compared to the visual situation. It means that they tend to make an analytical relationship and solved by formula or simple calculating procedure. For example, the area is 1 minus the area of the two outside triangles or the area of hexagon calculated by the formula using algebra relationships that arranged from the visual cases.

Visual thinking thought as a phenomenon that could introduce experimentally to a certain extent (Adler & Davis, 2006). In this research, that is by verification when the students solve the problems. The thinking is the reflections of abstract concepts and symbols and used for developing an insight expected to produce an answer independently from the representations. Rif'at (2017a; 2018, p. 11) ensures that a visual perception based on geometric concepts that operated to mental imagery. Students can take information and retain a mathematics problem by arranging and schematizing their thinking in the representations manner. The representation supports both algebraic and visual for comprehension and creativity and to improve students' thinking in solving the problems. Sophocleous, et al. (2009) report that the visual model in problem-solving facilitates students' comprehension and creates solution-finding opportunities. For example, the students used a ruler when exploring Pythagorean, and get numerous system of linear equations to show the truth. That is referred to as attributes of figures and viewed as visual thinking.

Tall (1991) argues that visualization is more effective than conventional approaches in strengthening students' intuitions and facilitating the learning. Tall (1991) considers visualization as a tool that serves to attract students' attention by drawing geometric concepts and models with varying effects to implicate the presence of various mathematical systems and various spaces. That helps students acquire an abstraction and to improve their cognitive independence and productivity, and to ensure meaningful learning and retention of information.

Concerning the two representations thinking, there is an etymological sense of a concept. That is the thinking by drawing, tabling, making relationships, and testing, or justifying. The goals are for determining the available information, both abstract or the practical sense. In the thinking models, there is a mind mapping method-a series of abstractions that represented algebraically or visually. That is a relation between operations and the implicit mapping in the logical connection seen similar to individual representation.

The logical connection often aroused in solving a problem. That is, most of the mathematics problems presented with hidden or less information, so the students need any assumption about the relationship among the representation that constructed. The students' views that there is logical reasoning and needed to solve the problems in particular. For example, it is a problem presented visually, as depicted in Figure 2. The students ask to relate G_1 and G_2 based on the visual information.

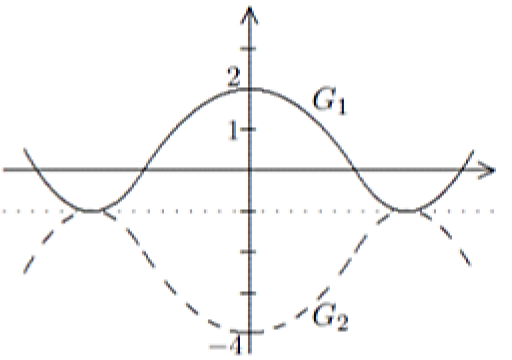


Figure 2. The G2 Function Constructed From G1 or Vice Verse

The logical connection is a composition of transformations and not to be done by the students. However, they need s strategy, for instance, visualize - G_1 (reflection by x-axis) and then turn down 2 units. They use algebraic thinking, i.e., by matrix transformation of some points in the dimension. The solutions are $G_2 = -G_1$ and $G_2 = - G_1 - 1$. Their algebraic thinking seems to organize algebraic expression. They also have a practice in using formulas or procedures. That is an operation in the matrix for getting a solution.

In the preliminary study, data collected from 25 students in geometry class and 39 of calculus. Invented that all of the 25 belief that $G_2 = (G_1)'$ or the derivative of G_1. The 39 students of calculus class solve it that $G_2 = -G_1$ (15 students) but one of them gives another visual illustration by simple graph of $y = x +1$ that reflected to x-axis and get that $y = - (x + 1)$; $G_2 = -$

$G_1 - 1$ (21 students) with two algebraic representations, i.e. $G_2 = -G_1 - 1$ and $G_2 = -(G_1 + 1)$; $G_2 = -G_1 + 1$ (1 student); $G_2 = G_1 - 1$ (1 students); and G_2 is equal to the inversion of $-(G + 1)$ from one other.

The students need to deal with simple geometrical representations and concepts rather than an arithmetic operation. From the study, it understood that students need a visual (the simplest) for constructing an equation. They want the equation to get another one according to a problem. For example, transforming the representation to a recognizable one but still not yet bring to the solution.

Another algebraic expression arranged by a matrix or transformation of two order matrix. It looks practical, i.e., only taking a point before and after the transformation. That is a linear transformation, and of course, the students come to an incorrect answer. It is a symbolic expression that precedes and leads to the intervention of the solution.

Methods

The researcher proposed solving mathematics problem abilities based on the students' empirical thinking. That ensures their abilities, skills, and knowledge in the representations and the effects of the activities - the observation based on verification of the representational thinking. The documentations are evidence-informed; replicable the effect student's progress toward achievement and influencing their performances of the representations.

The teaching activities are in geometry, algebra, trigonometry, and integrated into mathematics teaching and learning course. Student's performance measured to gain an understanding of the representations and the solving strategy arranged. The focus is on intervention, building-up of using the representation, and exploring the empirical thinking models. That is a meta-pattern categorized by type of representations according to thinking, a kind of performance spectrum in solving mathematics problems.

Research Design

The design is to verify solutions by considering the students' empirical thinking. In that case, the design recognized according to the steps by the representation. The empirical thinking is of verification, explored from the solution as explained in Figure 3.

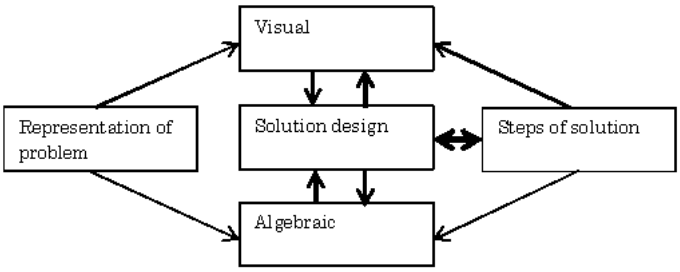


Figure 3. Diagram of Research Design

The research is an experimental design of teaching and learning. The design is based on students' solution linked from ways of thinking observed by the steps of solutions. The treatment controlled by the observed data trend about the significant of the representation oscillation in the answer. That is to verify fact of the thinking of the representations in the trend.

The trend was observed through steps of solution or the design of thinking. The observation managed through a deep discussion according to the same model of solution. The scope of study is pertained the using each of representation and the steps. That is an empirical change of mathematical thinking, i.e. power of logical connection and the rigor. The changing of the steps arranged in a pattern that describing a model of thinking.

Problem

The question that addressed is: What is the empirical verification thinking based on the visual and/or algebraic representation in the solutions of the mathematics problems? That is focus on the using of the representations to solve the problems, the different ways in visualization, the kinds of algebraic representation, and the thinking models. In other words, the problem of what a solution has provided considerable representation and the empirical verification thinking used by the students.

The models of thinking are dependent on the belief and experiences, so the development is not simply a logical implication of the thinking. An argumentation is in the context of the problems as an ability to integrate the situation into representational thinking. That is a thinking construction built imagination from the representation of the problems and how the visual and the algebraic come to the solutions.

The researcher gave students a written test to ascertain their solutions. They work in class. The test contained 5 questions and conducted in five times during a semester. The materials are basic mathematics that to be thought in class. The students use and manage information in using and extending representation of the problems for a solution. It is to enhance their abilities to solve mathematics problems where the knowledge viewed as integrated representations or in the case of the partiality.

Hypothesis

The empirical verification thinking has a trend by series of steps of the solutions according to algebraic and visual representations. The steps are in using the representations to get answer. And, the solution is in a meta-pattern of thinking.

Subject

Data collected five times at 3 different classes, odd semester in academic year 2018/2019. The distribution of the problem depicted in Table 2.

Table 2. Distribution of Problems in academic year 2018/2019

Problem	Type of information	Semester	Sum
1	Need a visual	1	23
2	Visual representation	1	14
3	Visual representation	3	37
4	Geometry knowledge	5	18
5	Spatial	7	20

All of the students are already get mathematics contents in the course, i.e. between 9 to 70 credits. The mathematics credits contained geometry, set theory and logic, and introduction of calculus. Much of the content is school mathematics and some are advanced material. In fact, the subject matter is competition content, i.e. for Olympiad.

The problems are about empirical thinking from algebraic and visual representations. The level is competitive, that is of reasoning and problem-solving. In this research, the two levels of competency that mean as thinking and the strategy. The visual thinking mainly based on the representation that used in solving the problems. The algebraic thinking was based on symbolic manipulation of the problems. But, there is also possibility to solve the problems used the representations simultaneously or alternately.

Procedure

The procedure of research is in constructing the models of empirical verification thinking from written test. The trend qualitatively includes documentation of practices to have the reasonable relationship that influence the future performance in solving mathematics problems. That means of carry out the representations as maximal as possible (Rif'at, 2017b). That is a model of thinking as depicted in Table 2.

Table 2. The Distribution of Using Representation Based On Thinking Models

Algebraic	Visual	Relation	Thinking Models
Symbolic	Geometry figure	Equation	Based on algebra
Equation system	Manipulating	Modeling	Visual thinking
Changing	Modifying	Manipulating	Algebraic thinking
Another equation	Adding	Complex	Complex algebraic

To construct the models, the researcher separates the representations, make a relation and the thinking model. In algebraic representation, there are symbols, system of equation, changing of the representation, and arousing another equation. In the visual, the students' activities are in the geometric figure, the visual manipulation, the modification, and for adding other visual when solving the problems.

The relations seem at equation constructed, the model of solution, by manipulating the representation, and the constructed complex relation. The representation and the relations show the empirical thinking by verification. The verification comes from the algebraic and the visual, and the arousing of complex representation, particularly in the algebraic. After classifying solutions, data arranged in accordance to the formulation of the students' thinking models of their empirical.

The thinking models of the empirical verification comprise of the power of algebra, the thinking until the complex one, and of visual thinking. The thinking can be detected from steps of the solutions, i.e. in alternating the representations, in manipulating or modifying, and in constructing them to get rational solutions.

The researcher and the students play an active role in the teaching and learning process. The researcher facilitates the learning and comprehension to represent the solutions algebraically and or visually. The approach is pedagogical, to gain the researcher' view and idea of the research. The teaching refers to the strategy that relies on explicit and implicit representations through lectures and the solutions. The teaching strategy is the ability-centered approach in using the representations in consistence.

Data Analysis

The growth and the modeling is both a variable and analyzed linear and nonlinear statistical technique for identifying patterns of growth of the representations. That is used to identify the development in the cognitive of the representations as an attempt to solve the problems. The data was mainly designed on the basis of mathematical representations, which provides contextual situations to address issues of the consistency of cognitive thinking. That is in the boundary or oscillation of using representations and the trend. That is the growth of the rationality of the pattern of the solution as thinking model.

To generate data, the representations are qualitative case, which used clinical interventions (project approach), discussion, and document analysis. That is an investigation of the students' solutions and the representations. The data is based on representation used of how big the consistence in the steps of the solutions as an understanding and insight of empirical thinking. The intervention and discussion sought to examine the students' experiences of the representation in solving mathematics problems and their views of the thinking context. The project questions directed to the level of representational thinking of the consistent change, to facilitate continue solution and to raise issues of grasping the representations and reflect on the experiences in solving the problems.

In line with the research questions, two major categories were used to process the data: (1) students' idea about representation thinking prescribed in the solutions, and (2) the ways in which the students recognize representational difficulties in the solving, how to help the students overcome those difficulties. Thus, a range of the solutions emerged relating to students' beliefs in learning mathematics, the contexts that reveal challenges to build thinking in the representation, and researcher efforts to preserve the way of thinking that is on the representation empirically. The emerging representations were compared across the alternating cases and the cross-cutting were identified, findings were categorized and steps of solution were drawn from the analysis and interpretation by the lack of the steps.

The researcher recorded and analyzed the visual and algebra thinking of the representations by verification of the empirical solutions. The models of thinking explained in **Figure 6**.

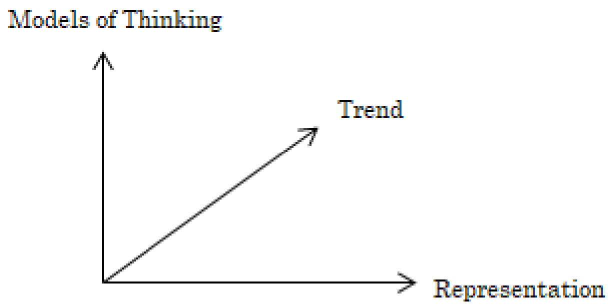


Figure 6. A Graphical Model of Empirical Verification Thinking

The representation axis shows the types (visual or algebraic) used by the students and the trend shows how consistence they are in using the representation. The consistency or how big the representation used is a model of solution that described in a graph and show trend. The trend is the model of thinking that describing a meta-cognitive type.

Results and Discussion

Result

Problem 1

The students ask to prove that: If α and β are two acute angles and α < β, then sin α < sin β.

Twenty students visualize right triangles (13 students) and 7 of them draw any triangle and then construct the heights. And, there 3 students are not to visualize it. The thirteen students use particular measure of the two acute angles, i.e. 30 and 60 degrees for α and β respectively.

They give the value of each of the function and the compare it. The students did not count to get the value, but just recalling from the previous learning. That is the solution.

Seven students use symbols for taking the ratio of sin α and sin β, but not used in steps of a solution. They give the ratio and compare it by inequality or analytic. One of the solutions is explained in Figure 3.

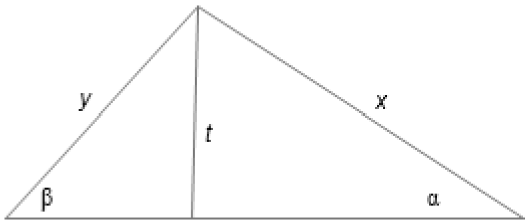


Figure 3. Visualization of Problem 1 Constructed by the Students

The ratio is $\frac{t}{x}$ and $\frac{t}{y}$ for *sin* α and *sin* β respectively. The inequality written by the conclusion, i.e. the students show first that $\frac{t}{x} < \frac{t}{y}$ and then they give the reason. The reason is because of *x* > *y* by perception. They look at the visual and state that *x* > *y* with measure it.

One student makes the same visual, but different logic used in the solution. That is: from the visual x > y so α < β. The logic connection is incorrect. The student uses the visual, but not to solve the problem. The student firstly describes the two sinus functions by ratio from the visual and then concluded that α < β based on the visual. After translating into English, the answer or solution is: "Could be seen at the visual that x > y with $\sin\alpha = \frac{t}{x}$ and $\sin\beta = \frac{t}{y}$ then α < β."

Another solution from three students is without visualization. The solutions are the same as depicted in Table 4.

Table 4. Steps of the Solution by Contradiction

Statement	Conclusion
Suppose that *sin* α > *sin* β	α > β
α > β	Contradiction to the hypothesis
α < β	*sin* α < *sin* β

The solution by contradiction (a type of doing proof) shows that the algebraic representation without the visual look like no guidance. They conclude α > β in step 1 only because of the equivalence of implication (contra positive), but still not to prove. And, the next step also can't bring to the rationality of logic. That is not a proof, as one of algebraic thinking problem. The logic statement in proof is for describing the equivalence.

Another type of problem 1 is to prove: if α < β and both are two acute angles, then $\frac{\sin\alpha}{\alpha} < \frac{\sin\beta}{\beta}$. There are some 'complicated' solutions. There are 18 students solving the problem. The types of the solution as depicted in Table 5.

Table 5. Types of the Solution by the Students Algebraically

Steps	Type 1	Type 2	Type 3	Type 4
1	$\frac{\sin\alpha}{\alpha}=1$	$\frac{\sin\alpha}{2\cos\alpha}<\frac{\sin\beta}{2\cos\beta}$	$\frac{\tan\alpha}{\alpha\tan\alpha}<\frac{\tan\beta}{\beta\tan\beta}$	$\frac{\sin\alpha}{\alpha\cos\alpha}<\frac{\sin\beta}{\beta\cos\beta}$
2	$\frac{1}{\cos\alpha}<\frac{1}{\cos\beta}$	$\frac{\sin\alpha}{\alpha}=1$	$\alpha<\beta$	$\frac{\sin\alpha}{\alpha}=1$
3	$\frac{\cos\beta}{\cos\alpha}<1$	$\frac{\cos\beta}{\cos\alpha}<1$	$\frac{\tan\alpha}{\alpha}<\frac{\tan\beta}{\beta}$	$\frac{1}{\cos\alpha}<\frac{1}{\cos\beta}$
4	$\alpha<\beta$	$\alpha<\beta$		$\frac{\cos\beta}{\cos\alpha}<2$
5	$\frac{\sin\alpha}{\alpha}<\frac{\sin\beta}{\beta}$	$\frac{\tan\alpha}{\alpha}<\frac{\tan\beta}{\beta}$		

Two types of the solution are in 5 steps, and other two are 3 and 4 steps. During class discussion was understood that all students used algebraic representation because of last experience. The logical connection of the algebraic representation is not a proof of the expression. For example, 5 students write that: suppose that $\sin\alpha>\sin\beta$ then α > β. Contradiction to the antecedent, so the statement is true.

When the researcher needs the detail and ask the students for explanation, they said: "if $\sin\alpha>\sin\beta$ then in a right triangle shows that α > β." That is a problem of doing proof of algebraic (or analytic) representation. That is an equivalent statement that usually used when the students considered the simplest one for elaborating or proving, but not a contradiction way.

An interesting respond is of using visual (geometric shape) representation. They draw a right triangle, i.e. the two angles are in one triangle. In that case, α > β so based on their perception concluded that $\sin\alpha>\sin\beta$. Why do they not use the same proof from the original statement? Most of the students said that one way to prove is by contradiction.

Problem 2

A square is divided into rectangular triangles as shown in Figure 4. The students ask to find the tan of β.

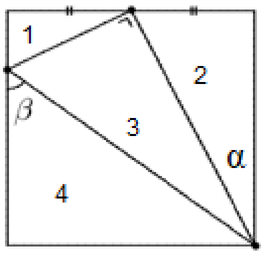


Figure 4. Complete Visualization by Symbols and Area Number of Problem 2

There are 14 students participated in solving the problem. Ten students take visual number 4 of Figure 4 to elaborate the situation to the solution. Tree of them work in the original picture. The picture number 4 used by the ten to solve the problem. The visualization explained in Figure 5.

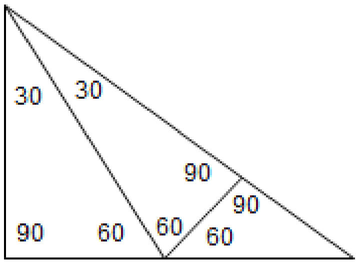


Figure 5. Part of the Original Picture Constructed by the Students

The students get a part of the visual of number 4 and then drawing it outside the original, and complete it as explained in Figure 5. There is no information of the completed by the measures. From discussion, the students belief that β = 30 + 30 = 600. Their beliefs based on measurement by protractor. They did not directed to ratio of tan, but construct other line segments to get magnitude of the angles. In general, students said that they have problems because no enough information at the problem.

Working from the original picture, three students respectively state that: β = 180-(30+90) = 600, starting from tan α to find α by calculator equal to 26.54 but can't get β and tan β, and the last one student' answer is $\alpha = \frac{1}{3}$ of right angle and equal to 30 degree. The conclusion is that β = 600.

After making a discussion or clinical investigation understood that mind mapping of the students are to look for particular triangles. That is their experiences during learning. But, that is image when meeting a visual representation. Other intact students answer the problem algebraically, i.e. using Pythagorean and the practical understanding. In general, there are two types the solutions, but more algebraically than visual empirical verification. They have not yet used visual representation, and not focus on the visual illustration.

Problem 3

Starting with a square of side 1, a regular hexagon is constructed, concentric with the square as in Figure 6. The students ask to find the area of the intersection of the both figures.

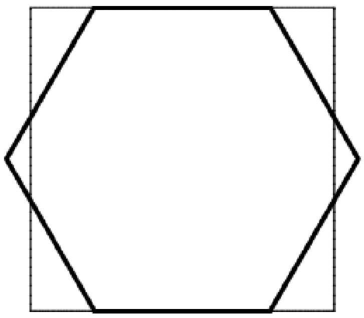


Figure 6. A Square of Side One Intersected with a Regular Hexagon

There are 37 students of semester 2 become participants in solving the problem. At amount of 31 students start their solutions, using area formula of a hexagon with a variable of the side. Sixteen of the students end their solutions with the variable in square, not find a number. The steps of the answers are full arithmetic works. That is a relation in algebraic but not as well as the representation, means that just simple relation. For examples: area of the intersection is 1.598 s2 where s is the length of the hexagon side (10 students), i.e. the area is equal to area of hexagon minus the square; the intersection area is $A = \frac{3}{2}\sqrt{3}\, s^2$ and $s = \left(\frac{1-2x}{6}\right)^2$ where x is a variable of the square side not of the hexagon (4 students); and calculating area of one equidistance triangle in the hexagon using sin function, multiplied by 6 and get the final answer is $s^2 = 2\sqrt{2}\, a^2$ where s = a, and the hexagon area, i.e. $\frac{3}{2}a^2\sqrt{3}$ where a is side of the hexagon. Fifteen of other students give answer that the area is: $\frac{1}{6}\sqrt{3}$ where the length of hexagon side is 1/3 and without minus the two small figures out-side the square (4 students); negative, i.e. $\frac{4-3\sqrt{3}}{8}$ that is equal to area of square minus the hexagon (1 student); the length of hexagon side is 2/3 and the answer is bigger than 1 (5 students); and the very big number of the answer because the students determined that the hexagon side is 9 (6 students).

Six students give different answers look more complicated by algebraic thinking and the relations in the representation. That is no adding information from the visual but the students give many numbers in the solutions. The answers are: the intersection area is $\left(\frac{3\sqrt{3}}{2}\right)(1-2a)^2$ where 1- 2a is the length of hexagon side; divide the visual into 2 trapezoids and a rectangular; using diagonals and conclude that the area of hexagon is equal to area of the square; using Pythagorean to get the hexagon side by the equation $a^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{a}{2}\right)^2$, and the inter-section area is $\frac{\sqrt{3}}{2}$; and count the area of 6 triangles outside the intersection using assumption that the two of the outside square is equal to the two triangles inside the square, so the intersection area is $x^2 - ab$, where x is side of the square, a and b respectively are the right side of the four triangles inside the square.

All of the answers are algebraic representations in their relations without any logic in the visual situation. There are some visuals made by the students, but not in relation to the question. They can work in arithmetic skills but the visual look like for information of the algebraic thinking. The visuals are two different visual made by the students. The intersection is not correct, some others of the students put the hexagon inside the square, and determined 6 triangles for getting an answer.

Problem 4

The students ask to find the cosine of the top angle α of one of the lateral faces as depicted in Figure 7.

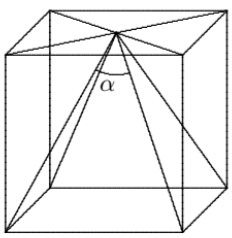


Figure 7. A Cube with a Constructed Regular Pyramid

There are 18 students of semester 5 solved the problem. Seven students change the visual as depicted in Figure 8.

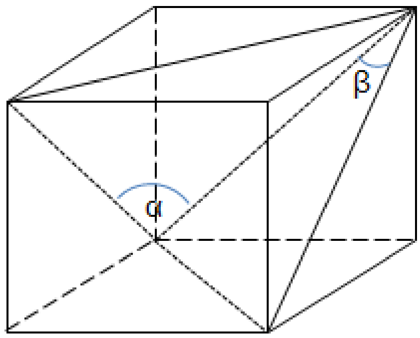
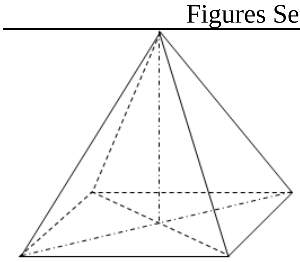
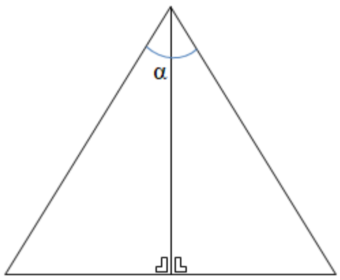
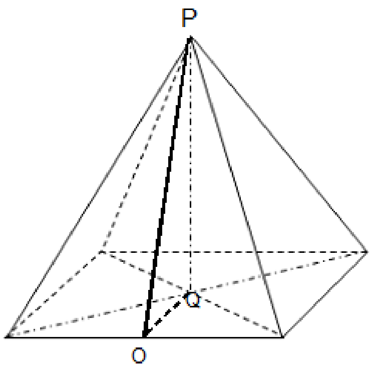


Figure 8. New Construction After Understanding Problem Situation

After constructing the figure, six students write that angle α at Figure 7 is same as at Figure 8. It looks no relation, but the students think that triangle at Figure 7 is equidistance, so the angle is 60 degrees. The students try to bring the problem to the area of the triangle to get α. One student redraws Figure 7 without diagonals of the top plane. She draws a net of pyramid inside the cube and concludes that four triangles of the net are equidistance, so α is 60 degrees.

Eight students take the pyramid out of the cube and think it. The types of the solution were depicted in Table 6.

Table 6. The Sequence of Figures and the Solution

Figures Sequence	Types of Solution
	1. The students' belief that the pyramid sides construct particular or equidistance triangles, i.e. α is 60 degrees. 2. The students declare that the particular triangles are isosceles.
	1. Two right triangles where the perpendicular line divides α into two same angles with α equal to 120 degrees. 2. The right triangle where the perpendicular line is the pyramid' height, and get the acute angle of the base using *cosines*, where the *cosines* is $\frac{1}{2}\sqrt{2}$ and the angle is 45 degrees. So, the triangle is particular with angles are 45, 45, and 90 degrees.
	1. Starting from right triangle PQO at Q, by Pythagorean and OQ = ½ PQ (?), the students get OP = $2\sqrt{5}$ and

The results assign to the models of thinking within each cell of Table 2. The researcher used a classifying construction as depicted in Table 3.

Table 3. The Performance of the Representations Based On Thinking Models

Thinking Models	Algebraic Representation	Visual Representation
Simplification	Using the formula	Based on the visual
Process	Separated from the algebra	Algebraically
Consistency	Changed to the different one	Alternately
Completion	Visual manipulation	Algebraically
Perception	Visually	Using the algebra
Insight	Developed to the algebraic	Algebra algorithmic

Each of the solutions analyzed in Table 3. That is according to the degree of the representations used in the steps of the solutions. That is the flow of the students' empirical thinking to complete the solution or get an answer. The empirical thinking verified from the solutions and short class discussion during the research.

The visual and algebra representations that used are mainly by formulas. The students use a formula in geometry and then algebraic. The algebra manipulation is from the relation but without the visual. That is a model of simplification. In the discussion, the students say that the visual representation helps them to memorize the formula and then solve it algebraically. In the process of the solution, the students back to the visual representation, to separate the algebra manipulation for getting another relation in the representation and then working algebraically. That is a type the thinking process.

The consistency of the thinking looks at an effort to get another visual built from the original used alternately in the solution. The students draw some visuals added to the original, but the solution forward divergently styles. It looks at the same adequate thinking between the two representations for a solution. The completion of the students' thinking is for the algebra relation. When they face the variables of an equation, the students try to get more visual representation to complete a comparison to the system. In the visual image, the students solve the problems algebraically.

Perception colored on the students' solution. Visual perception is for getting a solution but recognizable previously. They consider the visual representation to get algebra relations that possibly solved. So, in the algebraic representation, the students' perception is visually but using algebra in the solution. That is the insight into the steps of the solutions based on algebra. They develop their thinking patterns to the algebra and the algorithmic.

Discussion

Mathematical representation is rooted in the thinking gap. The ultimate experience or shared learning of thinking seems to hinder students from constructing representation connections. Due to the gaps, the students solve the problems with obstacles of the two representations. The positions of the representations show separate thinking. For the algebraic (or analytic), they face to their knowledge of the thinking in mathematic representations. Another style of thinking is of indirect competency of the visual representation or logic of geometry-the style linked to the ultimate experience becoming the representation for getting an analytic image and algebraic thinking.

The students demonstrate awareness about the algebraic but not from the visual construction. That is a challenge or at least an effect on mathematics learning. When the researcher gives the more visual situation of a problem, the 'eureka' motivates the thinking in visual representation. They try out their unique visual representation and the strategies to engage meaningful solutions by them. That is such a communicative style.

The solutions empirically depend on the pedagogical, i.e., using examples, convergent questioning, analogies, some cues and probing, and prompting the consistent steps. That is to fill mathematics holistically. The pedagogical are strategies grounded in the students' learning experiences, personal or group of students, particularly in resulting images to help them for making connections between algebra and the visual and vice versa.

The research treatment brings an elaborate understanding of what does it to learn in-depth. That is to elaborate students' conception of the representations comparing to the traditional (or algebraic thinking) and argue a transition to a student-oriented mode. For example, to move away from algebraic-based to an insight that sustained changes in the routine solutions. That is an activity-based by constructing the visual thinking focused on the representation.

While elaborating on the relation between the representations and the solutions, the "models of mathematical thinking" are still in the students' minds. They begin to present representations practically through the use of analytics. The student can master the process of the solving through the ultimate experience, i.e., the algebraic of the visual or vice versa. That leads to the development of students' mental capacity to engage the thinking at the level of semi-abstraction. The experiences can't enhance their motivation and encourages them to participate in the course actively. The researcher goes on to the consistent of thinking in every representation by stimulating and constructing the thinking.

If the routine teaching practices reflect creativity in using the representations not just alternately but also thinking in one representation consistently, the students can reshape their thinking capacity involved in meaningful learning. That is to put mental imagery effort into the courses to make the solution exciting and dynamic. A class environment facilitates students' thinking activity in solving problems independently.

The simplification thinking models tackled the solutions in a simple way. The representations involve opportunities to understand about solutions and thinking. The opportunities are to master the process of solutions to deeper understanding of the concept algebra and geometry. Their experiences are critical moment to grasp representational in the way of understanding mathematics.

It constructs the students' thinking models and helps them in recognizing and grasping the solution steps involved in the models. Reflecting on Problem 3, the algebraic manipulation: (1) the equations of the areas of the six small triangles constructed by the students' perception; (2) using formula of area of the hexagonal with supposing that the side is one unit, but avoiding two areas of small triangles outside the square; (3) using more than one variables in one equation to count the area of the six small triangles; (4) taking an equation that the area of the square is the same as the hexagon by the formulas and getting that area of the intersection is $\frac{9}{2}\sqrt{3}$ or $\frac{3}{2}\sqrt{3}$; and (5) by supposing that $(1-2x)$ is the side of the hexagon where x is a part of square side from the same two parts of one unit and come to the equation of the hexagon area is $\frac{3}{2}\sqrt{3}(1-4x+4x^2)$ without a solution.

Others solutions: (1) using a formula of diagonals of hexagon, i.e., $\frac{1}{2}\left(n(n-3)\right)$ as area of the hexagon where n = 6; (2) starting with an equation:

$$The\ area\ of\ hexagon = \frac{the\ area\ of\ the\ square + the\ area\ of\ the\ hexagon - 2 \times the\ area\ of\ the\ two\ small\ triangles\ outside\ the\ square}{2}$$

but no answer of the small triangle; and (3) constructing two trapezoids inside the hexagon and one rectangular and formed an octagon, and the height x of the trapezoids counted by proportion that $\frac{x}{a} = \frac{1}{\sqrt{2}}$ where x is the height and a constant particular size determined that is then a=$x\sqrt{2}$.

The students getting tried to construct the visual as an experience that come from feeding schools. Later, they understood it to use algebraic relations to give the formula. The students overcome the problems to simplify them. The reflective accounts embedded in the class experiences provide insights into the students' beliefs and the solutions geared towards mathematics learning. An analysis of the accounts reveals insights, ideas, and thinking activity that are relevant in understanding teaching for representations inconsistent ways.

Appears of an intended National Curriculum of Indonesia and what ought to be the course of mathematics in department of education. Realization of the curricular goals directed to the development of representational thinking, knowing and using geometry, and constructing new ideas in solving the problems. That provides visual representations of the problems situations in the learning, try to help students overcome the gap of representation thinking. As reflected in the students' types of answering, they were not able to obtain the solution needed more empirical thinking.

The nature of the students' differs in representation cases. The difficulty to solve is because of the two representations attributed to varied reasons. The analysis shows that the everyday experience caused by the imperfect mathematical knowledge of geometry concepts, particularly related to construct and to apply the law of the eternity of area. The students being unaware of their thinking required to perform the actual relations involved in solving the problems, but more prescribed in the textbooks. They also lacked in understanding algebraic manipulations (equations) to the situations. The students seem to have 'circling thinking' in the processes while solving the problems.

The result of the empirical verification thinking as a range of the research theme emerged relating to students' beliefs and performances of learning mathematics. The contexts revealed challenges to think in algebra and geometry, the nature of the challenging representation ideas and the efforts to build the students' thinking consistently. The emerging compared across the representational models and the thinking, identified the empirical types and were drawn and the interpretations.

Conclusion and Suggestion

This research highlights that students' mathematical thinking, particularly in using algebraic and visual representations, evolves through empirical verification processes driven by problem-solving activities. While algebraic thinking often dominates due to its procedural familiarity, students also exhibit various levels of representational thinking—sometimes mixing visual and symbolic forms irregularly, reflecting their cognitive flexibility and challenges. Visual representations serve not only as tools for understanding geometric properties but also as pathways for imagining and constructing solutions. However, these representations are often underutilized or disconnected from symbolic reasoning, leading to inconsistencies and incomplete solutions. The study reveals that effective mathematical learning should integrate visual and algebraic thinking through pedagogical strategies that emphasize meta-cognitive development, representation-based learning, and student-teacher relationships. By fostering such integration, students can better construct, manipulate, and communicate mathematical ideas, enabling them to solve problems meaningfully and develop a more comprehensive mathematical identity.

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