Developing of doing proof for school mathematics through teaching and learning

Mohamad Rif'at, Sudiansyah, Septian Peterianus
[1,2,3] Universitas Tanjungpura Pontianak, Indonesia
✉ mohammad.rifat@fkip.untan.ac.id

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Abstract

Understanding of concepts is a fundamental ability to learn mathematics in a more meaningful way. Understanding concepts is one of the goals to be achieved in learning mathematics. This study aims to determine the impact of a realistic mathematics learning approach (RME) and analytical thinking on students' understanding of mathematical concepts. This experimental research uses a posttest-only control design. The population in this study included students XI in Kasui sub-district, Way Kanan district, Lampung, Determination of the sample using cluster random sampling technique. The research sample was 72 students (36 in the experimental class and 36 in the control class). Hypothesis testing using ancova test. Based on the research results, it can be concluded that the realistic mathematics learning approach (RME) has an influence on students' understanding of mathematical concepts by controlling analytical thinking, analytical thinking has an influence on students' understanding of mathematical concepts, and there is a simultaneous influence between the RME approach and analytical thinking on understanding mathematical concepts student.

INTRODUCTION

The word and the etymological sense are two hints about effective use of language, about exploration techniques, and about developing representation of information. For example, the length and width of a rectangle are not about position, but a measurement or a quantity concept. Based on the example, an area problem which given a picture and the measurements is not mathematics problem. That is a language trajectory of representations or a problem of the etymological sense. When asked to find the perimeter of rectangle, but the two quantities are whole numbers and neither is divisible by 6, then, that is mathematics trajectory. However, if the area known, say 36 squares of a measure unit, then problem of getting the perimeter is to test or to check. That is also a kind of the etymological sense of meaning (Mohamad Rif'at, 2016, 2017a, 2018a, 2019).

A complex challenge in doing proof is to propose, identify, and clarify the thinking mathematics proof developed by taking account of diverse psychology, methodology, and culture context. One of the intersection domains is an etymological sense to study the students' doing proof to practice mathematics educational design. The etymological-based approaches for the development are to examine the relationships between multiple common language contexts appeared by diverse styles of teaching and learning such as education background, academic qualification, textbook used, higher education cultural, and curriculum in Indonesia (Mohamad Rif'at, 1998). Based on factors of the etymological, this research proposed an approach related to the methodology to investigate the doing proof as mathematics ability development.

It is true that some students are better at proof than others and also the vast majority of the students are capable of doing proof of school mathematics. Proving does not come as naturally as learning to prove, but have the meaning that usually is the definitive one. Many conceptions of mathematical proof have been confined to the formal and forms of the deductive. This research of adaptive proving is much broader, including not only informal explanation and justification but also intuitive and inductive based on the term of etymological sense. The formal proof requires meaningful expressions formed by logical connectives. The interpretation is determined by stipulating objects according to the standard interpretation of the logical. But, it is hard to separate out significant findings in logic from those in etymological sense because of the logic (Mohamad Rif'at, 2018b; Mohamad Rif'at, et al., 2019). That is mathematics system has an intended interpretation, whereas the logical leaves the possible one. For example, the truth in a formal system is true in some particular interpretation. So, the action in doing proof requires the valid view point to a mathematics statement.

Another example is about representation, visual or algebra. The teaching culture mainly based on the algebra where the symbols colored a proof. When the students face a visual representation, they tend to change or translate to algebra relationships. Founded that, the proof is often difficult be right or less of meaning. The researcher experience during 30 years in teaching always faces with the problem and sometime illogical in arranging the solutions. According to etymological sense of meaning, this research based on the science of languages refers to the pragmatics. That explores what students think to work using a word. For example, to test, what will they show or do with a mathematics statement, using a language of formal proof or the daily activity? While in a class, the lecturers make the word in the same meaning with to proof that implemented as usual as.

Students' behavior that observed is to follow the lecturers and the textbook. That is an indexical expression to incorporate reference to the importance and the interesting in doing proof of mathematics. But, in reality, the aspects have been instrumental in the development of the approach to philosophy commonly associated with the label of the word or sentence. For example, 'to verify' based on an experimental thinking, where the students understand the word with a trial activity by identifying, or comparing, or testing. That is more useful than rigidly wrote a proof using a particular way. When the researcher asks the students to proof that the sum of a triangle angles is equal to 1800, they verify it by an experiment. By step wise, they construct the others triangles from the original one and the experiment is to relate the changes from a triangle to another one. They get the relationships that the statement is true. The other students show the truth by combining two the same triangles by transforming or cloning. A word 'cloning' is the same as to know under what conditions it would be true. According to Mohamad Rif'at (2017b), the role of meaning in natural languages and the relation of ontology are receiving extensive consideration and discussion when proving a mathematics statement, because many students have doubted, however the efforts have been fruitful.

The researcher founded the inter-connected between words (or phrases) and the mathematics symbols (or notations). That is, the used of superlative words to the symbols or vice versa. The problem mainly based on the words than the symbols. For example, 'at least' or 'more than' or 'at a time, least than or equal to' are not so easy represented symbolically. When changed from a symbol to a word, there is no the rich word but less than or bigger than. That is a problem of the mathematics learning. Common usage of a word 'proof' often disturbed students in doing proof. For example, from a text book or lecturers' note founded an exercise: prove that the points (-3, 4), (6, -5), (-9, 10), and (21,-20) lye in a line segment. The data collected show the algebraic performance of the students, too many relationships of the symbols, and look difficult. But, when a word 'to prove' changed with 'to investigate', there are many activities conducted by the students. That shows a relation of a learning culture with habit.

There are many etymological meanings of 'proving' (Ernest, 1991; Leder, 1994), i.e., upright, forward, go through, try, test, judge, find valid, verify, and check. So, proving is a process of a state and part of improving thinking (Ball, Thames & Phelps, 2008). That is another kind of problem solving as a manageable craft (Candy, 1991). In the thinking configuration, a belief is very important to adopt a strategy for setting off-on a proof, i.e. to have an action of what the students do. The etymological sense puts the doing of proof as an ability to find analogical correspondences of words in reasoning. The analogical are tools to think with, serving as sources of hypotheses, of problem-solving operations and techniques, and to learn and to transfer information. When asked to prove a mathematical statement, students present evidence of answering and inference by taking samples. That is an experiment learning design, helping them to build representations by experiences, and demonstrating 'proving' abilities. For example, the students prove that $\sqrt{2}$ is an irrational number by testing or visualizing to a truth. The research suggests students able to present proving using words recognizable to motivate and to justify mathematics work. According to Wertheimer (1959), "to justify is in the sense of provide sufficient reason for." It means that proof is a form of justification, even though not all justifications are proofs. Wu (1999) states that, students can learn by justifying their mathematics ideas in proving as the earlier of the learning.

A tool for obtaining models of doing proof based on a construction of the same statement. If the given structures are models of etymological sense, then their representation is a model of thinking, because a mathematics statement is true in the technical sense employed. For example, to develop a representation model by evaluating an equation $x + y = 3$ is in R, also the existence of a proof. That is nice when many discussions of it. Mathematics educational terminology of proof is somewhat misleading and often leads to a question "How to teach or learn proofs?" But, an existing of a proof also written in prose, using a way of taste and convention of learn. That is a manageable task for mastering the art of proofs. So, the real question is "How to teach (learn) proving?" Hanna (1987) identified successful teaching strategies as requiring an organized approach to teaching. According to Hanna, the formal proof was taught until it was mastered. But, in a proving, the argumentation has three modes of available information: (a) the given; (b) the soliciting; and (c) the feedback. That requires a formal approach and the clarity of presentation and sequencing of information. While, Cooney (1985); Pehkonen (1997); Raymond & Leinenbach (2000); and Reid (1997) investigate that students actions within their reactions to what occurs in the classroom and identified as important determinants of teaching. The sequence of information and the feedback is better when presented in words than symbol (Mohamad Rif'at, 1998). That is the determinants of this research by exploring many words in action to prove and to get the mathematics ideas in reality.

METHODS

This study employs a qualitative longitudinal research design to examine the growth trajectories of students' mathematical proof abilities within the Master Teacher Program during the academic years 2019–2020. The research investigates how students construct and develop their proof skills over time, focusing on the role of etymological word usage in mathematical reasoning. The oscillation model of learning is used to map students' proof development across different stages, including thinking, drawing, doing, testing, and developing.

The study involves 30 selected participants from an initial pool of 100 master teacher candidates enrolled in a mathematics education program in 2018–2019. These participants were chosen based on their consistency in proof construction and mathematical reasoning. All participants have prior teaching experience at middle and high school levels, with some also engaged in private tutoring.

To collect data, the study utilizes mathematical proof tasks, structured observations, semi-structured interviews, and rubric-based assessments. The proof tasks assess students' reasoning and proof construction abilities, while structured observations document their mathematical terminology usage and logical structuring. Semi-structured interviews provide insights into their beliefs and perceptions about mathematical proof, and a proof evaluation rubric measures flexibility, originality, and elaboration in proof construction.

Data analysis is conducted using thematic analysis to identify common patterns in students' proof trajectories, comparative analysis to examine differences in proof construction based on etymological terms and logical reasoning, and trajectory mapping to categorize students' proof development into four quadrants: illustration, example-based proof, visual representation, and symbolic manipulation. Additionally, statistical techniques, including path analysis, are applied to determine relationships between students' proof abilities and their mathematical word usage. This methodological approach provides a comprehensive framework for understanding students' proof development and its implications for mathematical education.

RESULTS AND DISCUSSION

Results

The analyse procedure started from using words in the doing proof and ended in setting three creative aspects. It ended because the class had the cases, so seen as an interpretive plausibility. That included further consideration the models generated distinct patterns of growth proof trajectories by looking at their values of model fit criteria.

The entropy values indicated the creative clearest for proof models, showing that the three models were more appropriate than the one formal model. The plausibility suggested the three models tended to have the closest fit to the empirical data and interpretive in its growth proof trajectory patterns.

The mathematics teaching had significant formal proof capacities but with small effective in discriminating the students' performances. Compared with formal proof, the etymological sense of a proof was higher proofs degrees, positive class interactions, and the researcher' attention promoting student learning was more than the formal. The class had positive perceptions of mathematics teaching.

Student' beliefs to prove mathematics statements or expressions in the cross content exhibited strong needs to teach at high school. In decreasing order, the less like to become lecturer (teaching at higher education). The students were less likely to reduce their activities to prove in flexible, compared with rigid proof. They worked normally in the etymological sense of word meaning compared with the traditional because being engaged in fewer learning activities relative to the sensing.

Students' beliefs represented as reflections on difficult learning culture of doing proofs. They began experiencing the difficult before elaborating mathematics proofs by sensing word meaning. But then, there are differences in the difficult among using the meaning of words and increased during learning. They gradually diminished from formal proof, due to the interaction between student mathematics ability, the language forms (mixed application), and pursuing to prove. The class contained students who never experienced difficult in doing proof.

All of mathematics problems comprises of monophonic to context, and velocity to viscosity. There are four quadrant mapped student' performance into knowledge to information and vice versa, and information to data and vice versa. Student' knowledge is the same in general, but get it to an information of problems less than the knowledge. For example, in determining and relating an available knowledge to unknown information. That is a problem of formal proof.

When seeing from difference perspective of a proof, arousing a velocity to write down a proof mainly is in order to the formal. The velocity appeared in using numbers and operations considered to a clear solution. They get objective facts in a statement (or problem) as a monophonic viscosity to the context. For example, proving that if $x < 0$ then $x^2 > 0$, the students understand the smallest negative number is the key, i.e. based on line numbers. Another example is about absolute value. They always think right side of the line numbers.

The important is when passing Monophonic-Context axis, students presented data from a problem looking at the pattern. For absolute problem, they bring a number or an equation by transforming graph, a kind of action word. For example, the graph of $|2x-1|$ is $(2x – 1)$ for $x \geq ½$ or negative of $(2x – 1)$ or $(1 – 2x)$ for $x \leq ½$. For a negative number, there is a nice visual motion, it is from zero to the left and moved it also from zero to the right of a line number. That is a key that zero is the starting work of absolute concept, a viscosity thinking pattern.

The viscosity axis, the right of Monophonic-Context is a basis of doing proof. Student' level of the quadrant promoted complete thinking to a proof. An example of the thinking is that data in a mathematics problem showed according to their convincing. That based on some testing for a truth. For absolute values of $|2x-1|$, the solution set is $y \geq 0$, convincing the students that is the values.

Discussion

Mathematics proof trajectories reveal the etymological sense of meaning, trending to high-increase considered from the correct growth of solutions. The percentages are consistent with under graduate and high schools teacher. That is, large variations of performance, and identified the mathematics proof ability for students and teacher.

The extending of proof ability identified unique phenomena. The performance of formal proof generally remains an unchanged development or the growth tended to constant. That mainly used deductive approach through abstraction. The performance has a slight decline, becoming similar to lecturers and many textbooks. But, when investigating word direction of action, the performance observed in originality of the students' abilities.

All performances of doing proof exhibited an increasing trend with the initial appearing to elaborate flexible representation and determined with slight consistence (i.e. generally positive slopes). That is a unique pattern of the doing proof performance is also consistent with the sense of meaning (words in etymological) and relative stable. Examining the difference in measures reveals that the learning activities (experiment method), lecturer' notes, teaching and textbook can afford the students that seldom speaks or uses words of common language in proving.

Another unique phenomenon is that, the students exhibit a depression in their mathematics proof ability, which has not been observed in previous studies. The reasons explained by the results of meta-pattern analysis. The students in the high-increase performance in formal proof perceive positive the mathematics teaching, to pursue higher education (particularly in proving), and disengage from some activities in preparation for the etymological sense of word meaning. Their experiences measured using numerous of symbols and logic, but not higher performance than the contextual measures of high-increase sense of meaning. These two unique phenomena need to be validated by future research.

Contexts of different proof trajectories, obtained an unbiased complete understanding of the etymological sense of words meaning. The results reveal meaningful: student' belief, mathematics teaching, textbook toward proof performance are in descending order. The high quality of formal proof, the less activities in proving based on textbook or teaching (lecturer' culture). The student' beliefs are still formal proof, and founded that they come to oscillation of presenting proof.

Students in the high-increase performance of doing proof tend to perceive high-quality mathematics teaching (in lecturing, interaction, and textbook), compared with the etymological sense of meaning performance suggests that teaching proof must relate to student' activities for gradually proving. A case is in plotting points, by symbol or visual in grid paper. The high-quality of proof started with equation, while etymological sense of meaning is from real perception and then constructing or inventing many concepts before given. That is positive perceptions similar to construct word trajectories. However, the survey methodology cannot confirm a cause-and-effect relationship, so the reasons for the results should be discussed from diverse perspectives or at least consider the possibility that mathematics teaching and proof ability affect one other.

Despite the generally high-quality mathematics proofs presented in master teacher education program in Indonesia (Rif'at, 1998, p. 27), the results cannot rule out the possibility that high-achieving students experience better mathematics teaching including by the etymological sense of meaning (especially for school math). If this is the case, inequality in mathematics proof teaching remains an issue for educators and important addressed to innovative measures for increasing educational equality continuously develop in Indonesia.

When considering the relationship in the opposite direction, there is a possibility that high-proof ability students tend to be more perceptive of high-quality mathematics teaching compared with other factors (textbook, belief, or formal representation). In other words, student perceptions of doing proof is irrelevant to mathematics teaching approaches, as indicated by the findings of previous research regarding the uncertain relationships between mathematics teaching and student mathematics ability (e.g. Byrnes & Miller, 2007).

The reasoning suggests a project of etymological sense of meaning in proving mathematics ideas onto educators' quality and implies that students' perceptions or assessments of learning or teaching biased to their ability naturally. Future research and master teacher educational program may need to consider student ability or achievements when assessing teaching quality, particularly in doing proof.

The view of student beliefs is that students in the high-increase in doing formal proof disengage from learning activities when preparing for examinations, but the etymological sense of meaning tend to feel satisfaction with mathematics proof. By contrast, the low-increase students have motivations to pursue high-performance proof, studying curriculum and enjoying proof activities; however, they experience frustration with mathematics proof early (starting from logic, abstraction, and deductive view point) and becoming exacerbated during the courses). To simplify this view, mathematics proof becomes similar with student' ability matches their study goals or curricula. Tailored curricula matching students' proof ability ought to be developed, the mater teacher educational design in Indonesia must reduce constraints of doing proof, and provide flexibility to accommodate students proving activities among personal abilities (etymological sense of meaning), textbook, lecturer' teaching culture, and master teacher educational program designs (see Chiu, 2016).

Limitations of this research suggested for future research according to the selection of empirical contextual measures, which provides a development of mathematics learning culture. The measures, however, was mainly designed on the basis of etymology and student' beliefs in doing proof, and related to mathematics learning. For example, a set of mathematics problem could be presented language based, not only symbolic.

So, future research needs to elaborate research findings focusing on mathematics education or experimental studies in scientific cultures. Many of the contextual measures were measured discretely (not so in order) or dichotomously. Self-reported measures using a Likert-scale sometimes can't provide information for in depth discussion. For example, it is difficult to express an inner idea of student' mind, and to examine a mathematics statement is also hard if without a logic.

This research uses an empirical condition as the data collection method, which cannot be used to make claims about cause-and-effect relationships. Although the contextual measures (e.g. mathematics teaching) imply causes for proof ability, only experimental designs can address a cause-and-effect relationship. Finally, the measures collected sufficient data in relation to proof teaching and learning that permeates Indonesian education. However, is a complicated issue of its interplay with contextual measures mapping on meta-pattern and future research could provide insights into issue of doing proof. For instance, in using representations, is it algebraically, visually or verbal.

The string of the etymological meaning of the proving of the school mathematics (*vertical-axis*) was the representation of a proof. These are composed of data, meaning, learning experiences, and the patterns. Data presented in a mathematics problem mainly information for thinking. That is a phrase and not enough for doing proof. For example, a test: On the initial side of angle θ, let P be a point one unit away from the vertex. If $\theta \geq 0$, let s be the arc length traced by P as the initial side rotates through θ; If θ is negative, then the arc length s negative, then the radian measure of θ is s. The students ask to describe it and to get an equation of degrees and radians.

That is an etymological sense of meaning. A word 'initial side' and 'trace' related to the concepts, and not easy to visualize it without an understanding of coordinate-system. Students need a meaning of the words, for instance linear or not for the 'trace' and the 'initial side' about position. Their experience must be developed so they have numerous elaboration of the test.

A word 'rotation' is also an etymological sense of the meaning. It is a problem of describing correctly the situation. Another word is 'negative or positive', that looks like a number. However, the words in that context are not just numbers, but also a direction. So, words or phrase in doing proof always have mathematical meanings, the learning is a main factor when teaching. For example, the test illustration made students to be difficult to construct an equation, because of the meaning of angle and 'trace'. That is also a problem of unit measure of the two words.

A completeness of proof is also problem of phrase or sentence. For example, "If θ is negative, then the arc length s negative, then the radian measure of θ is s." is that reversible or not? So, how about validity of logic? The contrapositive is not rigor, means no good experience for students learning.

The level of doing proof of the student was seen as an important mathematics performance, and hence a useful etymological sense of effective teaching. This indicated that the doing proof analysis was necessary. Figure 1 shows the results of this analysis. That described of the teaching and learning performance. The model accounted for all of the response. That increase is a substantive interpretation of doing proof etymologically.

Hierarchically, the etymological sense starting with data presented the proving. A range of student' level of doing proof was developed. The growth is in proving and the ability to prove. When they prove a statement, their understanding is to get a pattern of how to prove. Sometimes it is so hard because their experience influence it. So, they go back to the meaning of 'prove' by the etymological sense. For instance, state that to prove is the same as to illustrate.

The students tend to use data in the statement for proving, so their understanding decrease, i.e., in using logical connection. But, the visual representations help them for continuing their proof to get more meaning. In some cases, the students test the truth in every step, and some of them can develop the proof and come to specific pattern. But, in general their performance is in fluctuation.

The content of the etymological sense includes learning experience, belief, value, motivation, and information. The understanding part is the function of knowledge that provides a framework for incorporating the sense and information. The students relate back to the formal within an idea and evaluating the second idea. For example, to prove that a prime number is 2 or an odd is by taking 7. Another example is when proving that $|x||x| = x^2$ in R, the students use a specific number, but there is no logical connected or investigation by cases.

The findings show that the systems of elementary logic to be understood as the actual choice of the truth of a statement and assumed to be the correct one. A concept more general than validity is that of the relation of the statement or 'common sense' implication between a possibly of sentences and a single sentence that holds. This suggests a requirement on an informal system of truth. That is to reduce a formal representation to a finite case, and taken by a relativity of the completeness of the situation.

Knowledge is in an objective sense, consisting of the expression of proof problems. The data are unique to the current class environment designing by the researcher. The data comes too fast to try to do both proving and understanding at the same time. The context is weaving together or connection of words to act in the proof. That shapes student perception and interpretation of meaning. The information and knowledge comes to the students in a variety of word contexts. To understand or communicate meaning, the students attend to the contextual clues attached to each meaning. For example, proving a symbolic implication statement that $0< x < 1 \rightarrow 0 < x^2 < 1$, the students judge, determine, evaluate, and to consider the truth.

In the case, the used of 'approve' is also an enthusiastic representational thinking. The student' performance is to test and find the truth in the sense of validating once. But, after approving, mathematics induction less-used compared to an elaboration by solution design using the words. The student' solution is to evaluate the statement using a number bounded below or above of the interval to convince the truth.

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Patterns and context are closely related. The pattern tends to create its own context rather than being context dependent to the same extent that information is. The context normally comes explicitly, where the context is easily recognized, such as in a textbook. For the hidden context, information of a problem needs to identify. It is often extremely hard to verbalize properly and may not be constant. It is often the most difficult attribute for understanding a proof.

A proof presentation is how the data or information is arranged. Since this *organization* is conceptual, the presentation can create its own meaning or highlight it. When interpreting data, the students use the explicit one, such as keyword, visuals, and experience, and the presentation to form and shape the information.

A range of student' level of doing proof was developed. The growth is in proving and the ability to prove. There are some clear and consistent findings which are also supported by some of the survey result (reported by the class). The findings show that the students have more successful, to exercise in building a proof and considerable in developing mathematics teaching and learning programs adapted to the needs of students. The researcher in the case knew well and tailored their programs to meet their learning needs and stage of development in mathematical understanding.

The findings challenge some conventional views about the so-called 'axiomatic' of the school curriculum caused by subject-based. There was no evidence of incompatibility in the school math between effective teaching and learning. There have been considerable changes over the past thirty years in internal management structures and career structures within schools. In some sectors, these changes have often had negative effects on the status and responsibilities of subject in relation to other positions of responsibility within school math.

When the strings of information together and add complexity, the students form knowledge (sometimes a new one). The context has a beginning and an end of: (1) forming a meaning, and composed of parts of data; (2) starting to give meaning to the parts (data), arrange into a representation; (3) formatting the contexts and added to it (experience); and (4) joining to a pattern. For example, why that the statement $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ true? The models of the proofs or argumentations were presented in Table 1.

Table 1. Models of proof according to the etymological sense of the meaning

Solution Model	Etymological
$\frac{1}{2}-\frac{1}{3}=x\rightarrow 1-\frac{2}{3}=2x\rightarrow\frac{1}{3}=2x\rightarrow x=\frac{1}{6}$	To judge
$\frac{1}{2}+\frac{1}{2}=2\left(\frac{1}{3}+x\right)$	To try
$\frac{1}{2}=\frac{1}{3}+\frac{3x}{3}=\frac{1+3x}{3}$	To verify
$\frac{1}{2}+\frac{1}{3}=\left(\frac{3}{2}\times\frac{1}{3}\right)+\frac{1}{3}=\left(\frac{3}{2}+\frac{2}{2}\right)\times\frac{1}{3}=\frac{5}{2}\times\frac{1}{3}$	To determine
$\frac{1}{2}+\frac{1}{3}=\left(3\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\right)=(3+2)\times\frac{1}{6}$	To test

While it used to be true that becoming of a subject was a step carrying considerable brings of either in some school math. The findings give reason to reconsider this trend. The nature of the immediate work appears to play an important role in teaching and learning. In summary, the studies indicate that there could be value in giving attention to the role of etymological of mathematics proof and the adequacy of current methods for preparing educators for transition into the role of Subject Coordinator.

The performances met at the levels of teaching and learning, primarily for school math. The goals standards or expectations that the students have for products, the design that enables goals to be met efficiently learning, and the management that ensure goals are up-to-date and are achieved by the students. The combination of the levels is one model of teaching and learning proof concepts with the results in nine performance variables, as in Table 2.

Table 2. Level of teaching and learning of proof etymologically

Level of performance	Goals	Design	Management
Class	Strategy	Structure	Resources
Process	Developing	Improving	Reengineering
Individual	Coaching	Intervention	Elaborating

To manage the nine performance variables will lead to the quality of teaching and learning mathematics representatively. Every improvement viewed through the table. That is to provide diverse opportunities for exploring the etymological based on the meaning and to discover more proof representations. The is a sense in terms of experience. The activity is of sense making and proving. And, the students' performance affected by dissociation accounts of the reasoning.

The students' doing of proof should be an issue to assess attention to mathematics education, especially learning and teaching mathematics. For example, found that etymological sense in learning to prove have effect to the achievement. The attitude towards mathematics stems from the development of early learning, which influences the views of students about mathematics to become more mature. Unfortunately, it appears in the learning that students are still 'imitating from educators, friends themselves, or other sources' in learning mathematics.

CONCLUSIONS

The findings of this research highlight the crucial role of etymological sense of meaning in shaping students' mathematical proof abilities and development. The growth trajectories of proof construction are influenced by students' understanding of words, phrases, and sentences, which in turn impact their ability to logically connect and structure proofs. Students initially rely on informal representations, such as illustrations and examples, before gradually progressing towards symbolic manipulations and formal proofs. However, their proof abilities are not solely dependent on mathematical structures but are also deeply rooted in linguistic comprehension and contextual interpretation. The study underscores the necessity of teaching proof through accessible language, using terms like "to show," "to draw," "to test," and "to verify", which support students in bridging the gap between informal reasoning and rigorous mathematical proof. Moreover, students' learning culture and past experiences significantly influence their proof strategies, making contextual teaching and linguistic support essential in fostering proof competency. The research also reveals that 80% of students require practical proof exercises, while only 20% rely on formal proof approaches, indicating a strong need for interactive and language-driven teaching materials. Additionally, evaluation techniques should incorporate both individual and group assessments, balancing formal logic and etymological reasoning. Ultimately, this study emphasizes that proof literacy development should be a gradual process, integrating intuitive understanding with structured proof strategies, while ensuring that students' beliefs and proof competencies are reinforced through targeted instructional design.

REFERENCES

Ahmed, W., Van der Werf, G., Kuyper, H., & Minnaert, A. (2013). Emotions, self-regulated learning, and achievement in mathematics: A growth curve analysis. *Journal of Educational Psychology*, *105*, 150–161.

Aunola, K., Leskinen, E., Lerkkanen, M. K., & Nurmi, J. E. (2004). Developmental dynamics of math performance from preschool to Grade 2. *Journal of Educational Psychology, 96*, 699–713.

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *The Journal of Teacher Education. 59*(389).

Beckstead, J. W. (2012). Isolating and examining sources of suppression and multicollinearity in multiple linear regression. *Multivariate Behavioral Research*, *47*, 224–246.

Benson, R., & Brack, C. (2009). Developing the scholarship of teaching: what is the role of e-teaching and learning?. *Teaching in Higher Education, 14*, 71-80.

Birkeland, M. S., Melkevik, O., Holsen, I., & Wold, B. (2012). Trajectories of global self-esteem development during adolescence. *Journal of Adolescence*, *35*, 43–54.

Bleeker, M. M., & Jacobs, J. E. (2004). Achievement in math and science: Do mothers' beliefs matter 12 years later? *Journal of Educational Psychology, 96*, 97–109.

Bodovski, K., & Youn, M. J. (2010). Love, discipline and elementary school achievement: The role of family emotional climate. *Social Science Research, 39*, 585–595.

Bowers, A. J., & Sprott, R. (2012). Examining the multiple trajectories associated with dropping out of high school: A growth mixture model analysis. *Journal of Educational Research, 105*, 176–195.

Bronfenbrenner, U. (1986). Ecology of the family as a context for human development:

Research perspectives. *Developmental Psychology*, *22*, 723–742.
Byrnes, J. P. & Miller, D. C. (2007). The relative importance of predictors of math and science achievement: An opportunity–propensity analysis. *Contemporary Educational Psychology, 32*, 599–629.
Candy, P.C. (1991). Understanding individual nature of learning. *Journal of Self-direction for Lifelong Learning: A Comprehensive Guide to Theory and Practice*, 52(249).
Chiu, M.-S. (2012). Differential psychological processes underlying the skill-development model and self-enhancement model across mathematics and science in 28 countries. *International Journal of Science and Mathematics Education*, *10*, 611–642.
Chiu, M.-S. (2019). An ecological approach to adolescent mathematics ability development: Differences in demographics, parenting, mathematics teaching, and student behaviors and emotions. https://doi.org/10.1080/03055698.2019.1672522
Chiu, M.-S., & Whitebread, D. (2011). Taiwanese teachers' implementation of a new 'constructivist mathematics curriculum': How cognitive and affective issues are addressed. *International Journal of Educational Development, 31*, 196–206.
Cooney, T.J. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education*. 16(324).
Ernest, P. (1991). *The Philosophy of Mathematics*. Great Britain: Burges Science Press.
Gomez-Chacon, I. M. (2000). Affective influences in the knowledge of mathematics. *Educational Studies in Mathematics*, *43*, 149–168.
Hanna, G. (1987). Instructional strategies and achievement in grade 8, *Proceedings of 11th International Conference for the Psychology of Mathematics Education*. Montreal, Canada.
Junco, R. & Cotten, S. R. (2012). No A 4 U: The relationship between multitasking and academic performance. *Computers & Education, 59*, 505–514.
Lan, Y. J., Chang, K. E., & Chen, N. S. (2012). CoCAR: An online synchronous training model for empowering ICT capacity of teachers of Chinese as a foreign language. *Australasian Journal of Educational Technology*, *28*, 1020-1038.
Leder, G.C. (1994). Research in mathematics education-constraints on construction? In Bell G (Ed.) Challenges in mathematics education: Constraints on construction *Proceedings of the 17th Annual Conference of Mathematics Education Research Group of Australasia*, 1995. Lismore: MERGA.
Mason, J., Burton, L., & Stacey, K. (1996). *Thinking mathematically*. Essex: Addison-Wesley.
McLeod, D. B. (1994). Research on affect and mathematics learning in the JRME: 1970 to the present. *Journal for Research in Mathematics Education, 25*(6), 637–647.
Pehkonen, E. (1997). Teachers' conceptions of mathematics teaching *Paper presented at the Current State of Research on Mathematical Beliefs Conference*, 1998. Helsinki: University of Helsinki.
Peixoto, F., Sanches, C., Mata, L., & Monteiro, V. (2017). "How do you feel about math?": relationships between competence and value appraisals, achievement emotions and academic achievement. *European Journal of Psychology of Education*, *32*(3), 385–405.
Raymond, A. & Leinenbach, M. (2000). Collaborative Action Research on the Learning and Teaching of Algebra: A Story of One. *Journal of Educational Studies in Mathematics*. *41*(283).
Reid, D. (1997). Constraints and opportunities in teaching provin. *Paper presented at the 21st Annual meeting of the International Group for the Psychology of Mathematics Education*, 1998. Lahti: Finland.
Mohamad Rifat. (1998). Analisis Tingkat Deduksi dan Rigoritas Susunan Bukti Mahasiswa Program Studi Pendidikan Matematika IKIP Malang [Unpublished master thesis]. IKIP Malang.

Mohamad Rifat. (2017). Filsafat pendidikan matematika [Philosophy of mathematics education]. Pontianak: UNU-Press.

Mohamad Rifat. (2018). The exploring of visual imagery: In their relation to the students' mathematical identity. *Journal of Higher Education Research, 5*(75).

Mohamad Rifat, Rachmat Sahputra, Sugiatno, & Dede Suratman. (2019). A developing of analytic inequalities monograph based on students' mathematical activities. *International Journal of Innovative Studies in Science and Engineering Technology*, 5(5).

Vanhalst, J., Goossens, L., Luyckx, K., Scholte, R. H., & Engels, R. C. (2013). The development of loneliness from mid- to late adolescence: Trajectory classes, personality traits, and psychosocial functioning. *Journal of Adolescence, 36*, 1305–1312.

Werheimer, M. (1959). *Productive Thinking*. New York: Harper & Row.

Wu, H. (1999). Basic skills versus conceptual understanding: A bogus dichotomy in mathematics education. *The Journal of American Educator*. *14*(19).