Prime ideal on the endomorphism ring of $End_{\mathbb{Z}}(\mathbb{Z}^n)$

Zakaria Bani Ikhtiyar[1]*, Nikken Prima Puspita[1], Titi Udjiani[1]

[1] Departement of Mathematics, Universitas Diponegoro, Semarang, Indonesia.
✉ zakaria.bani45@gmail.com[*]

Abstract
The set of all endomorphisms over R-module M is a non-empty set denoted by $End_R(M)$. From $End_R(M)$, we can construct the ring of $End_R(M)$ over addition and composition function. The prime ideal is an ideal that satisfies the properties like prime numbers. In this paper, we take the ring of integer number $\mathbb{Z}$ and the module of $\mathbb{Z}^n$ over $\mathbb{Z}$ such that the $End_{\mathbb{Z}}(\mathbb{Z}^n)$ is a ring. Furthermore, we show the existence of the prime ideal on the $End_{\mathbb{Z}}(\mathbb{Z}^n)$. We also applied a prime ideal property to prime ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$.

INTRODUCTION

A module over a ring is a generalization structure of a vector space over a field (Matlis, 1968; Nobusawa, 1964; Wahyuni et al., 2016). A vector space requires a commutative group and a field. In module theory, a field can be replaced with any ring with unity (Marks, 2002; Volodin, 1971; Jensen & Lenzing, 1989). Furthermore, the concept of a linear transformation is known in a vector space. This concept is also implemented in modules called module homomorphism.

Let M and N be modules over a ring R. If $f: M \to N$ is a module homomorphism in which $M = N$, then we call f a module endomorphism (Nicholson, 1976). Here we collect all of the endomorphisms in a module M over ring R such that we have the set of endomorphisms M over R denoted by $End_R(M)$ (Lindo, 2017). The endomorphism set is non-empty because at least there is the identity function as an element of $End_R(M)$. Therefore, based on ring theory in Herstein (1975), by adding two binary operations, i.e., addition and composition function, we constructed a ring called an endomorphism ring, denoted by $(End_R(M), +, \circ)$.

In a ring R, a non-empty set $I \subseteq R$ that satisfies the axioms of the ring and $ri, ir \in I$ for all $r \in R$ and $i \in I$ is called ideal (Davvas, 2006; Jianming & Xueling, 2004). Furthermore, the special ideal was defined by Dedekind in 1871 based on the properties of prime numbers. The concept of the prime ideal is known (Kleiner, 1998). Moreover, the prime ideal was discussed by Khariani et al. (2014), Maulana et al. (2019), and Khairunnisa and Wardhana (2019). In this paper, we have a special ring, i.e., the endomorphism ring $(End_R(M), +, \circ)$ where $M = \mathbb{Z}^n$ and $R = \mathbb{Z}$. This research shows the existence of a prime ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$. We also applied a characteristic of prime ideal for the prime ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$. Here we start our discussion about the prime ideal by giving the fundamental theories of a module over a ring.

Definition 1. *(Wahyuni et al., 2016) Let $(R, +, \cdot)$ be a ring with unity, $(M, +)$ a commutative group, and the scalar operation $\star: R \times M \to M$. A group M is a left R-module if satisfy the following conditions:*
i. $r_1 \star (m_1 + m_2) = r_1 \star m_1 + r_1 \star m_2$,

ii. $(r_1 + r_2) \star m_1 = r_1 \star m_1 + r_2 \star m_1,$

iii. $(r_1 . r_2) \star m_1 = r_1 \star (r_2 \star m_1),$

iv. $1_R \star m_1 = m_1,$

Furthermore, the group M is the right R-module if satisfy the following conditions:
i. $(m_1 + m_2) \star r_1 = m_1 \star r_1 + m_2 \star r_1,$

ii. $m_1 \star (r_1 + r_2) = m_1 \star r_1 + m_1 \star r_2,$

iii. $m_1 \star (r_1 . r_2) = (m_1 \star r_1) \star r_2,$

iv. $m_1 \star 1_R = m_1,$
for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Furthermore, group M is called an R-module if it is both a left and right R-module. A group M, as a module over a ring R, can be written by R-module M.

Example 2. Let $\mathbb{Z}$ be a ring with unity, $\mathbb{Z}^n = \{[a_1 \ a_2 \ a_3 \ ... \ a_n]^T | a_i \in \mathbb{Z}, i = 1, 2, 3, ..., n\}$ be a commutative group with a scalar operation $\cdot : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{Z}^n$. Group $\mathbb{Z}^n$ is a module over $\mathbb{Z}$.

Let M, N be modules over R. We can construct a relation between both of them, and we called it a module homomorphism as follows:

Definition 3. *([Dummit and Foote, 2004](#)) Let $(M, +_M)$, $(N, +_N)$ be modules over ring R with scalar $\star_M$ and $\star_N$, respectively. Map $f: M \to N$ is called R-module homomorphism if satisfy:*

> *i.* $f(m_1 +_M m_2) = f(m_1) +_N f(m_2),$ *and*
>
> *ii.* $f(r \star_M m_1) = r \star_N f(m_1),$

for all $r \in R$ and $m_1, m_2 \in M$.

Example 4. Let $(2\mathbb{Z}, +)$ and $(\mathbb{Z}^n, +_1)$ be modules over $\mathbb{Z}$ with the scalar operation $\cdot$ and $\cdot_1$, respectively, and $+_1$ is an addition between two vectors in $\mathbb{Z}^n$. Defined $f: 2\mathbb{Z} \to \mathbb{Z}^n$ by $f(2n) = [2n \ 4n \ 6n \ ... \ 2n^2]^T$ for any $2n \in 2\mathbb{Z}$ and $[2n \ 4n \ 6n \ ... \ 2n^2]^T \in \mathbb{Z}^n$. Based on Definition 3, we have that for any $2n, 2m \in 2\mathbb{Z}$, and $k \in \mathbb{Z}$, this is satisfy:
> *i.* $f(2n + 2m) = f(2n) +_1 f(2m),$ *and*
>
> *ii.* $f(k \cdot 2n) = k \cdot_1 f(2n).$

By this fact, we have that f is a homomorphism $\mathbb{Z}$-module.

In Definition 3, when $M = N$, we can call f an R-module M endomorphism. We can collect all R-module M endomorphisms in a set denoted by $End_R(M)$. This paper will have some properties of the set of all $\mathbb{Z}$-module $\mathbb{Z}^n$ endomorphisms. According to the $End_\mathbb{Z}(\mathbb{Z}^n)$, we have the trivial example when we define $\vartheta: \mathbb{Z}^n \to \mathbb{Z}^n$, i.e., $\vartheta([x_1 \ x_2 \ x_3 \ ... \ x_n]^T) = [0 \ 0 \ 0 \ ... \ 0]^T$ with $\vartheta \in End_\mathbb{Z}(\mathbb{Z}^n)$.

For studying this paper, we need to know about the definition of ideal and ideal, which is generated by an element as follows:

Definition 5. *([Adkins & Weintraub, 1992](#)) Let R be a ring and a non-empty set $I \subseteq R$. The set I called a left ideal on R if it satisfies the following conditions:*
i. $i_1 - i_2 \in I,$ *and*

ii. $ri_1 \in I.$

Furthermore, I is called a right ideal on R if it satisfies the following conditions:

i. $i_1 - i_2 \in I$

ii. $i_1 r \in I$

for all $r \in R$ and $i_1, i_2 \in M$. The set I is called an ideal on R if it is both a left and a right ideal on R. For all $I \neq R$, then I is called a proper ideal on R. Furthermore, for an ideal whose elements are only zero elements, it is called a zero ideal, denoted by 0.

Definition 6. *(Gallian, 2017) Let R be a ring and $a \in R$. A denoted by $aL = ra | r \in R$ generates the left ideal on $\langle \ \rangle \{R\}$, and the right one on R is generated by $a\langle \ \rangle_R = \{ar | r \in R\}$. If $\langle a \rangle_L = \langle a \rangle_R$, then $\langle a \rangle$ is a notation for the ideal on R generated by a.*

Based on Definition 6, we have the following example.

Example 7. Let $\mathbb{Z}$ be a ring and $2 \in \mathbb{Z}$. We have $2\mathbb{Z} = \{2n | n \in \mathbb{Z}\}$ as an ideal on $\mathbb{Z}$ generated by 2, denoted by $\langle 2 \rangle$.

The property of prime number can be applied for ideal such that we have the following definition.

Definition 8. *(McCoy, 1949) Let R be a ring and P proper ideal on R. Ideal P is called prime ideal on R, if for all $a \ b \in R$, with $aRb \subseteq P$, then $a \in P$ or $b \in P$.*

Example 9. Let $\mathbb{Z}$ be a ring, and $2\mathbb{Z}$ proper ideal on $\mathbb{Z}$. Ideal $2\mathbb{Z}$ is prime ideal on $\mathbb{Z}$ since, for all, $a \ b \in \mathbb{Z}$ with $a\mathbb{Z}b \subseteq 2\mathbb{Z}$ always results in $a \in 2\mathbb{Z}$ or $b \in 2\mathbb{Z}$.

In addition to defining the prime ideal, McCoy (1949), in his article, gave the following lemma relating to the prime ideal.

Lemma 10. *(McCOy, 1949) If I an ideal on R and P is a prime ideal on R, then $I \cap P$ is the prime ideal on I.*

METHODS

To investigate this project, we need to survey relevant topics. The initial step is studying the concept of the prime ideal in the work of McCoy (1975) and Wahyuni et al. (2016). Next, we study the application of the prime ideal in particular rings of previous research by Khariani et al. (2014), Maulana et al. (2019), and Khairunnisa and Wardhana (2019). Then we analyze the module structure, module homomorphism, module endomorphism, and the endomorphism of ring $End_R(M)$. This paper determined and chose $R = \mathbb{Z}$ and $M = \mathbb{Z}^n$. The last step is giving an example of a prime ideal in $End_{\mathbb{Z}}(\mathbb{Z}^n)$ and a prime ideal property that apply to it.

RESULTS AND DISCUSSION

The set $End_{\mathbb{Z}}(\mathbb{Z}^n) = \{f : \mathbb{Z}^n \to \mathbb{Z}^n | f \ is \ a \ \mathbb{Z} - module \ homomorphism\}$. For every $[a_1 \ a_2 \ a_3 \dots a_n]^T, [b_1 \ b_2 \ b_3 \dots b_n]^T \in \mathbb{Z}^n$, $f \in End_{\mathbb{Z}}(\mathbb{Z}^n)$ can be written as $f([a_1 \ a_2 \ a_3 \dots a_n]^T) = [b_1 \ b_2 \ b_3 \dots b_n]^T$. Because the codomain entries are integers, the Greatest Common Divisor (GCD) can be found. For example, if the GCD of each entry in the codomain is k, that is $(b_1, b_2, b_3, \dots, b_n) = k$, then the function f can be written as $f([a_1 \ a_2 \ a_3 \dots a_n]^T) = k[c_1 \ c_2 \ c_3 \dots c_n]^T$ with $c_i = \frac{b_i}{k}$, $i = 1,2,3, \dots n$ and $[c_1 \ c_2 \ c_3 \dots c_n]^T \in \mathbb{Z}^n$. With this fact, we have the following proposition.

Proposition 11. Given set $End_{\mathbb{Z}}(\mathbb{Z}^n)$ and $f \in End_{\mathbb{Z}}(\mathbb{Z}^n)$ with $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = [b_1 \, b_2 \, b_3 \ldots b_n]^T$. Element f can be written as $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k[c_1 \, c_2 \, c_3 \ldots c_n]^T$ with $c_i = \frac{b_i}{k}, i = 1,2,3, \ldots n$ and $k = (b_1 \, b_2 \, b_3 \ldots b_n)$.

For $f, g \in End_{\mathbb{Z}}(\mathbb{Z})$ with $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k[c_1 \, c_2 \, c_3 \ldots c_n]^T$ and $g([a_1 \, a_2 \, a_3 \ldots a_n]^T) = l[d_1 \, d_2 \, d_3 \ldots d_n]^T$, we define that $(f \circ g)([a_1 \, a_2 \, a_3 \ldots a_n]^T) = kl[m_1 \, m_2 \, m_3 \ldots m_n]^T$.

It was known that $End_{\mathbb{Z}}(\mathbb{Z}^n)$ is a non-empty set. Based on ring theory by Gallian (2017) an Koh (1971), the set $End_{\mathbb{Z}}(\mathbb{Z}^n)$ with the binary operation addition "+" and composition "∘" function can generate a ring called the endomorphism ring of $\mathbb{Z}$-module $\mathbb{Z}^n$, denoted by the endomorphism ring of $End_{\mathbb{Z}}(\mathbb{Z}^n)$. Therefore based on Definition 6., an ideal in the ring $End_{\mathbb{Z}}(\mathbb{Z}^n)$, which is generated by f, can be formed as follows.

Example 12. Given ring $End_{\mathbb{Z}}(\mathbb{Z}^n)$ and $f \in End_{\mathbb{Z}}(\mathbb{Z}^n)$, with $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k[c_1 \, c_2 \, c_3 \ldots c_n]^T$. The right ideal generated by f, i.e.
$$\langle f \rangle_R = \{f \circ g | g \in End_{\mathbb{Z}}(\mathbb{Z}^n), g([a_1 \, a_2 \, a_3 \ldots a_n]^T) = l[x_1 \, x_2 \, x_3 \ldots x_n]^T\}$$

$$= \{h | h([a_1 \, a_2 \, a_3 \ldots a_n]^T) = kl[d_1 \, d_2 \, d_3 \ldots d_n]^T\}.$$

Based on Example 12., the prime ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$ is explained in the following proposition.

Proposition 13. *Given the ring $End_{\mathbb{Z}}(\mathbb{Z}^n)$, $f \in End_{\mathbb{Z}}(\mathbb{Z}^n)$, with $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k[c_1 \, c_2 \, c_3 \ldots c_n]^T$. If k is a prime number, then*
$$\langle f \rangle_R = \{f \circ g | g \in End_{\mathbb{Z}}(\mathbb{Z}^n), g([a_1 \, a_2 \, a_3 \ldots a_n]^T) = l[x_1 \, x_2 \, x_3 \ldots x_n]^T\}$$

$$= \{h | h([a_1 \, a_2 \, a_3 \ldots a_n]^T) = kl[d_1 \, d_2 \, d_3 \ldots d_n]^T\}.$$

Is prime ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$.

Proof:
We have $End_{\mathbb{Z}}(\mathbb{Z}^n)$, $f \in End_{\mathbb{Z}}(\mathbb{Z}^n)$, with $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k[c_1 \, c_2 \, c_3 \ldots c_n]^T$. It will be shown that if k is the prime number, then $\langle f \rangle_R$ is the prime ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$.
Taken $\theta, \gamma \in End_{\mathbb{Z}}(\mathbb{Z}^n)$ with $\theta([a_1 \, a_2 \, a_3 \ldots a_n]^T) = r[x_1 \, x_2 \, x_3 \ldots x_n]^T$, $\gamma([a_1 \, a_2 \, a_3 \ldots a_n]^T) = s[y_1 \, y_2 \, y_3 \ldots y_n]^T$, and $\theta \circ End_{\mathbb{Z}}(\mathbb{Z}^n) \circ \gamma \subseteq \langle f \rangle_R$. Since "∘" it is a competition function, we get

$$End_{\mathbb{Z}}(\mathbb{Z}^n) \circ \gamma = \{g \circ \gamma | g \in End_{\mathbb{Z}}(\mathbb{Z}^n), g([a_1 \, a_2 \, a_3 \ldots a_n]^T) = l[m_1 \, m_2 \, m_3 \ldots m_n]^T\}$$

$$= \{\omega | \omega([a_1 \, a_2 \, a_3 \ldots a_n]^T) = ls[d_1 \, d_2 \, d_3 \ldots d_n]^T\}.$$

Next, we have

$$\theta \circ End_{\mathbb{Z}}(\mathbb{Z}^n) \circ \gamma = \left\{ \theta \circ g \circ \gamma \, \middle| \, \begin{matrix} g \in End_{\mathbb{Z}}(\mathbb{Z}^n), g([a_1 \, a_2 \, a_3 \ldots a_n]^T) \\ = l[m_1 \, m_2 \, m_3 \ldots m_n]^T \end{matrix} \right\}$$

$$= \{\eta | \eta([a_1 \, a_2 \, a_3 \ldots a_n]^T) = rls[d_1 \, d_2 \, d_3 \ldots d_n]^T\}.$$

Since $\theta \circ End_{\mathbb{Z}}(\mathbb{Z}^n) \circ \gamma \subseteq \langle f \rangle_R$, then rls is a multiple of k, in other words, $rls = km$, with $m \in \mathbb{N}$. Since s is the GCD of the codomain entries of any function $f \in End_{\mathbb{Z}}(\mathbb{Z}^n)$, then s is not always a prime number, so it is not always a multiple of k. Since k is prime, this results in

$r = vk$ or $s = vk$ for some $v \in N$. This means $\theta \in \langle f \rangle_R$ or $\gamma \in \langle f \rangle_R$. Based on Definition 8, $\langle f \rangle_R$ is a prime ideal in ring $End_{\mathbb{Z}}(\mathbb{Z}^n)$. ∎

Before being given prime ideal property on $End_{\mathbb{Z}}(\mathbb{Z}^n)$, additional conditions will be given to Lemma 10. In Lemma 10, no special conditions are given for ideal I and prime ideal P. This condition has the possibility that $I \cap P = I$. Consequently, there is the possibility that the ideal $I \cap P$ is not a proper ideal in I. This means that the condition of a prime ideal based on Definition 8, which must be a proper ideal, is not satisfied. Therefore, given the following lemma to complete Lemma 10.

Proposition 14. *If I ideal on R, P prime ideal on R, and I $\nsubseteq$ P, then I $\cap$ P prime ideal on I.*

Proof:
We have the *I ideal on R, P prime ideal on R, and I $\nsubseteq$ P*. It will be shown that *the I $\cap$ P prime ideal on I*.

First, it is necessary to show that $I \cap P$ is ideal in I. Take any $a, b \in I \cap P$, and $r \in I$. This means $a, b \in I$, and $a, b \in P$. Since I and P are ideal in R, then $-b \in I$ and $-b \in P$, so $a - b \in I$ and $a - b \in P$. This shows that $a - b \in I \cap P$. Furthermore, because I is ideal, then $ra \in I$ or $ar \in I$. So it is proven that $I \cap P$ is left and right ideal in I. Since $I \nsubseteq P$, then $I \cap P$ is a proper ideal in I. To show that $I \cap P$ prime ideal in I, take any $a, b \in I$ with $aIb \subseteq I \cap P$. It will be shown that $a \in I \cap P$ or $b \in I \cap P$.

Therefore $aIb \subseteq I \cap P$, then $aRbRb \subseteq aIb \subseteq P$. Therefore $aRbRbR \subseteq P$. Since P is a prime ideal, then $aR \subseteq P$ or $bR \subseteq P$. If $aR \subseteq P$, then $aRa \subseteq P$, since P is a prime ideal, then $a \in P$. This also applies if $bR \subseteq P$, then $b \in P$. Therefore it means $a \in I \cap P$ or $b \in I \cap P$. So that I$\cap$P is the prime ideal of I. ∎

Furthermore, based on Proposition 14, we claim the following proposition as the prime ideal property on $End_{\mathbb{Z}}(\mathbb{Z}^n)$.

Proposition 15. *Given ring $End_{\mathbb{Z}}(\mathbb{Z}^n)$, $f_1, f_2 \in End_{\mathbb{Z}}(\mathbb{Z}^n)$ with $f_1([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k_1[c_1 \, c_2 \, c_3 \ldots c_n]^T$, $f_2([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k_2[c_1 \, c_2 \, c_3 \ldots c_n]^T$,*

$$I = \langle f_1 \rangle_{R_1} = \{f \circ g \mid g \in End_{\mathbb{Z}}(\mathbb{Z}^n), g([a_1 \, a_2 \, a_3 \ldots a_n]^T) = r[x_1 \, x_2 \, x_3 \ldots x_n]^T\}$$

$$= \{h \mid h([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k_1 r[b_1 \, b_2 \, b_3 \ldots b_n]^T\}, \text{ and}$$

$$P = \langle f_2 \rangle_{R_2} = \{f \circ g \mid g \in End_{\mathbb{Z}}(\mathbb{Z}^n), g([a_1 \, a_2 \, a_3 \ldots a_n]^T) = s[x_1 \, x_2 \, x_3 \ldots x_n]^T\}$$

$$= \{h \mid h([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k_2 s[b_1 \, b_2 \, b_3 \ldots b_n]^T\}.$$

For all k_1 composite numbers, k_2 prime numbers, and k_1 not multiples of k_2, then I $\cap$ P is prime ideal on I.

Proof:
Based on Proposition 13., we have that I is ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$, but not prime ideal since k_1 is a composite number, and P is prime ideal on $End_{\mathbb{Z}}(\mathbb{Z}^n)$ since k_2 is a prime number. Furthermore, since $r \neq ms$ for $m \in \mathbb{N}$, then $I \nsubseteq P$. So, based on Lemma 13., $I \cap P$ is a prime ideal in I.

∎

CONCLUSIONS
The right ideal constructed by any $f \in End_{\mathbb{Z}}(\mathbb{Z}^n)$ where $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k[c_1 \, c_2 \, c_3 \ldots c_n]^T$, i.e., $\langle f \rangle_R$, is a prime ideal if k prime. If k is a composite number, then $\langle f \rangle_R$

is not a prime ideal. Furthermore, the condition for an intersection between an ideal and a prime ideal in a ring holds that the intersection is a prime ideal in an ideal. It must be an ideal constituent set, not a subset of a prime ideal forming set. The last result is that the intersection between the ideals formed by $f_1([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k_1[c_1 \, c_2 \, c_3 \ldots c_n]^T$ and $f_2([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k_2[c_1 \, c_2 \, c_3 \ldots c_n]^T$ with k_1 composite numbers and k_2 prime, is a prime ideal on the constructed ideal $f_1([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k_1[c_1 \, c_2 \, c_3 \ldots c_n]^T$.

For extension of our research, we can change $f([a_1 \, a_2 \, a_3 \ldots a_n]^T) = k[c_1 \, c_2 \, c_3 \ldots c_n]^T$

by $f\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}\right) = \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$. Then the properties of the diagonal matrix can be used

to examine other properties of the ring $End_{\mathbb{Z}}(\mathbb{Z}^n)$ and the prime ideal in that ring.

AUTHOR CONTRIBUTIONS STATEMENT
ZBI designs research and writing script. NPP conducts project investigations and contribute to interpretation results. TU designed the study and reviewed it script. All authors read and approve the final manuscript.

REFERENCES

Adkins, W. A., & Weintraub, S. H. (1992). *Algebra : An Approach via module theory*. Springer-Verlag.

Davvaz, B. (2006). (∈,∈∨ q)-fuzzy subnear-rings and ideals. *Soft Computing*, *10*(3), 206-211.

Dummit, D. S., & Foote, R. M. (2004). *Abstract algebra* (3rd ed.). John Wiley & Sons.

Fraleigh J B. (1999). *A first course in abstract algebra*. Addison Wesley Publishing Company.

Gallian, J. A. (2017). *Contemporary abstract algebra* (9th ed.). Cengange Learning.

Groenewald N. (2020). Weakly prime and weakly completely prime ideals of noncommutative rings. *International Electronic Journal of Algebra, 28*, 43-60.

Herstein I.N. (1975). *Topics in algebra* (2nd ed.). John Wiley & Sons.

Jensen, C. U., & Lenzing, H. (1989). *Model theoretic algebra with particular emphasis on fields, rings, modules* (Vol. 2). CRC Press.

Jianming, Z., & Xueling, M. (2004). Intuitionistic fuzzy ideals of near-rings. *Scientae Math Japonicae*, *61*(2), 219-223.

Khairunnisa, Y., & Wardhana, I. G. A. W. (2019). Sifat-sifat ideal prima pada gelanggang noether. *3rd ELPSA Conference*, 60–64.

Khariani, Q., Amir, K. A., & Erawaty, N. (2014). Ideal prima dan ideal maksimal pada gelanggang polinomial. *Jurnal Matematika, Statistika, & Komputasi*, *11*(1), 71–76.

Kleiner, I. (1998). From numbers to rings: The early history of ring theory. *Elemente Der Mathematik*, *53*, 18–35.

Koh K. (1971). On one sided ideals of a prime type. *Proceeding of the American Mathematical Society*, *28*(2), 321-329.

Lindo, H. (2017). Trace ideals and centers of endomorphism rings of modules over commutative rings. *Journal of Algebra*, *482*, 102-130.

Marks, G. (2002). Reversible and symmetric rings. *Journal of Pure and Applied Algebra*, *174*(3), 311-318.

Matlis, E. (1958). Injective modules over Noetherian rings. *Pacific Journal of Mathematics*, *8*(3), 511-528.

Maulana, F., Wardhana, I. G. A. W., & Switrayni, N. W. (2019). Ekivalensi ideal hampir prima dan ideal prima pada bilangan bulat gauss. *Eigen Mathematics Journal*, *2*(1), 1–5.

McCoy, N. H. (1949). Prime ideals in general rings. *American Journal of Mathematics*, *71*(4), 823–833.

Nicholson, W. K. (1976). Semiregular modules and rings. *Canadian Journal of Mathematics*, *28*(5), 1105-1120.

Nobusawa, N. (1964). On a generalization of the ring theory. *Osaka Journal of Mathematics*, *1*(1), 81-89.

Volodin, I. A. (1971). Algebraic K-theory as extraordinary homology theory on the category of associative rings with unity. *Mathematics of the USSR-Izvestiya*, *5*(4), 859.

Wahyuni, S., Wijayanti, I. E., Yuwaningsih, D. A., & Hartanto, A. D. (2016). *Teori ring dan modul*. Gadjah Mada University Press.