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Systematic study of the parabola with the contribution of GeoGebra software as a teaching proposal

Renata Teófilo de Sousa[1*], Francisco Régis Vieira Alves[1], Maria José Araújo Souza[2]

[1] Postgraduate Department in Science and Mathematics Teaching, Federal Institute of Education, Science and Technology of Ceará, Brazil.
[2] Undergraduate Department of Mathematics, State University Vale do Acaraú, Brazil.
✉ rtsnaty@gmail.com

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Abstract

This work aims to present different demonstrations of the parabola, as well as possibilities of its geometric construction, using geometric design techniques and the GeoGebra dynamic geometry software. The methodology of this work is a basic theoretical research, exploratory type, in which we seek to bring a view about the parabola focused on improving its teaching as mathematical knowledge with the contribution of GeoGebra software. As a result, we bring a set of five constructions made in GeoGebra and available for use, which can be used as a methodological resource by the teacher to work in the classroom. As this work is part of an ongoing master's research, as future perspectives, we aim to develop these constructions in the classroom and collect empirical data for further analysis and discussion.

INTRODUCTION

Analytical Geometry is a branch of Mathematics that brings a relationship between Geometry and Algebra, in which we can analyze algebraic questions geometrically and vice versa (Sousa & Alves, 2022). In the specific case of parabolas, these are commonly studied in the school stage in a way related to the graph of a quadratic function. However, its study within the field of Analytical Geometry is usually dissociated from the study of functions and even from its origin as a conic section, the reality of many Brazilian schools (Cerqueira, 2015; Macedo, 2015; Siqueira, 2016).

In this way, several methods for the construction of a graphic representation of the parabola stem from algebraic procedures, which are sometimes insufficient for the interpretation and understanding of its geometric elements (Halberstadt, 2015; Lucena & Gitirana, 2016; Sousa & Alves, 2022). This fact makes it difficult to learn through the prism of Analytical Geometry.

As a result of this fact, as the author Halberstadt (2015) points out, it is greatly difficult for the student to understand graphically what the parabola equation represents and the relationship between its variables to solve problems. In addition, it is also common not to connect topics that deal with the study of the parabola, such as the quadratic function, the parabola equation itself in its general and canonical form in Analytical Geometry, the representation of the parabola in the polar system and the parabola as the section of a cone.

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Sometimes these topics of study are taught as different topics in the classroom without making the connection between them.

According to Sousa and Alves (2022), an approach that links the different ways of studying the parabola is little explored in the classroom, causing difficulties for the student when entering higher-level courses in the area of exact sciences and coming across disciplines that demand this knowledge. Louzada (2013) adds that "in the case of the parabola, if the teacher is not careful, the student may have the conception of the cyclic definition that this curve in the plane is the graph of the quadratic function and that the graph of the quadratic function is a parabola," thus disregarding other forms of visualization and interpretation of this curve. The authors Lucena and Gitirana (2016) point out that the study of the parabola takes place through "a teaching that prioritizes the algebraic treatment and leaves the geometric interpretation of the represented mathematical object to be desired, which makes it difficult to go deeper into the concepts in focus." This made us reflect on the importance of seeking different approaches to teaching this subject aimed at both high school and higher education classrooms and promoting dialogue between algebra and the geometry of the parabola. Thus, this work aims to present different demonstrations of the parabola, as well as possibilities for its geometric construction, using geometric design techniques and the GeoGebra dynamic geometry software. In this way, we seek to broaden the discussion of the topic and facilitate the student's understanding of geometric visualization, considering that this relationship is little discussed in textbooks.

Conics have been studied for a long time, and their main definition was deduced in the masterpiece of Apollonius of Perga (262-190 BC) called "The Conics," which earned this mathematician the title of "The Great Geometer" (Eves, 2011; Boyer, 2012). However, the most famous applications of conics are due to Galileo (1564-1640), who in 1604 concluded that the trajectory of a cannonball describes a parabola, as well as Kepler (1571-1630) and Newton (1643-1630). 1727), who found through their research that the orbits of the planets are elliptical.

Thus, conics are mathematical objects studied with a brief approach in High School and their extension in undergraduate courses at a higher level, usually in Analytical Geometry. In general, the definition of a conic is given from two points of view: the intersection of a plane with a cone and as a locus of points, which are divided into circumference, parabola, ellipse, and hyperbola. However, in this work, our interest is directed only to the conic parabola.

Bermúdez and Mesa (2018) state that conics - ellipse, parabola, and hyperbola - are generally presented in textbooks as a didactic unit, and their demonstration assumes that students have a prior basis on their mathematical concept. This can be seen in the way the content is introduced in the classroom and in the way in which its definitions are arrived at, from the generalization of concepts, without carrying out a previous survey that allows having some judgments or basic geometric elements in the construction and obtaining of the equation that represents the curve of a conic.

Alves and Pereira (2016) emphasize that the language presented in textbooks and used in the teaching of Analytical Geometry itself has flaws, generating difficulties in understanding the subject because the way the topic is approached treats its topics in a dissociated way, sometimes not relating these topics to previous knowledge and not exploring the real meaning of each term involved. However, an efficient understanding of these concepts is necessary for the student's cognitive development in the Geometry field, not only in High School but also in Higher Education.

In this sense, the teacher must have professional competence developed in the epistemic and didactic areas, in which he must seek ways for clear transmission of the content with real mathematical meaning, possibilities of applications, as well as better resourcefulness in his performance, reflecting on the understanding and learning of its students.

Regarding the teaching of the topic of parabolas within Analytical Geometry, there are few applications seen in the classroom, both in High School and Higher Education in Brazil (Bermúdez & Mesa, 2018; Siqueira, 2016; Macedo, 2015; Halberstadt, 2013; Cerqueira, 2015), which reverberates in the student's difficulty when faced with some higher-level disciplines such as Analytical Vector Geometry, Linear Algebra and Differential, and Integral Calculus, for example.

Based on the above, we understand that there are gaps in the training of students in both High School and Higher Education regarding the study of the parable. Thus, we seek to develop in this research a proposal to integrate different ways of visualizing the parabola geometrically through technology, more specifically, the GeoGebra Dynamic Geometry software. GeoGebra is an open-access software that allows manipulations in a Dynamic Geometry computational environment, defended by many authors in different research worldwide. About understanding the world around us from geometric visualization, Alves (2019) states that with the GeoGebra software, students can develop a capacity for global and local analysis of properties extracted from the computational and geometric environment.

The GeoGebra environment allows the visualization and manipulation of its elements and constructions. Alves and Borges Neto (2012) state that exploring GeoGebra as a technological instrument allows the visualization of unimaginable situations when restricted to pencil and paper. The same author also reiterates that based on the potential of the GeoGebra software, the teacher, when using it, can stimulate student involvement in a dynamic exploration of numerical and geometric properties, developing visualization, perception, and intuition, essential for the evolution of student learning (Alves, 2019).

METHODS

The methodology of this work is basic theoretical research of the exploratory type because, according to Prodanov and Freitas (2013), this research model aims to generate new and useful knowledge for the advancement of science, but without necessarily an expected practical application. In this way, our intention in this work is to bring a view about the parable focused on improving its teaching as mathematical knowledge. Thus, we treat this study as intending to increase our scientific knowledge base. In Figure 1, we have the flowchart research:

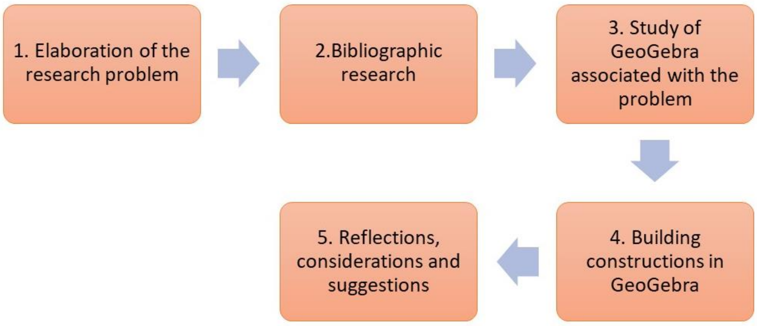

Figure 1. Flowchart research

Structurally, we have:

- *Research design:* theoretical bibliographic research;
- *Participants:* not applicable;
- *Instruments:* GeoGebra software and study materials (books, dissertations, and articles published in scientific journals);
- *Data analysis:* qualitative, exploratory type.

To organize this research, we structured the text from a bibliographic survey on the characteristics of teaching parabolas in Brazilian schools and the potential of the GeoGebra software as a resource to carry out its teaching through different types of mathematical visualization of the parabola.

This research was carried out based on the study of nineteen materials, including books, dissertations, and articles published in scientific journals. The subjects researched revolve around the themes: of conics, parabolas, GeoGebra, the history of mathematics, and the use of technologies in mathematics teaching. The results were obtained from the reading of the materials, seeking a deeper view of the teaching of the parable in the classroom in the Brazilian context, in which we developed a didactic proposal for its teaching to facilitate the learning of the theme, with an alternative methodology to work with the parabola using the geometric design and the contribution of the GeoGebra software.

Therefore, our research follows the following path:

(i) Theoretical framework: we bring a bibliographic survey on the teaching of parables and its difficulties, and we suggest GeoGebra as a resource for its teaching;
(ii) Results: we suggest different demonstrations and geometric views about the parabola, serving as a didactic-methodological proposal using GeoGebra;
(iii) Our considerations and recommendations from this study.

Given the fact that this research is being developed in the Postgraduate Program in Science and Mathematics Teaching (PPGECM, acronym in Portuguese) of the Federal Institute of Education, Science, and Technology of Ceará, in Brazil, as part of a study in the master's course, data collection and analysis will be carried out in later work.

RESULTS AND DISCUSSION

Based on the theoretical framework studied to develop this research, we observed that different ways of presenting the parable are not presented in the books and materials used in the classrooms of Brazilian schools. There is also a lack of teaching on the subject of technology. So, we developed different constructions in GeoGebra, presented here as a result of this study and support for the mathematics teacher.

In this section, we seek to bring different ways of building a parabola and exploring its characteristics, with didactic suggestions from manual geometric drawing techniques to the possibilities with the GeoGebra dynamic geometry software, in order to facilitate teaching and the student's geometric visualization.

- **The parable, through paper folds.**

The steps for understanding this construction are, respectively:

1. When we take a square sheet of paper (preferably), we must draw a line d and mark a point F inside it, not belonging to line d;
2. From this, choose any point belonging to the line d and fold the sheet so that the chosen point coincides with point F;
3. When unfolding the sheet, we must have creases produced by the folding done;
4. By repeating this process several times, considering as many points as possible, we will see that all the creases marked on the paper will form the figure of a parabola with focus F and directrix d, corresponding to the line drawn on the paper initially;

We can see this construction exemplified in Figure 2:

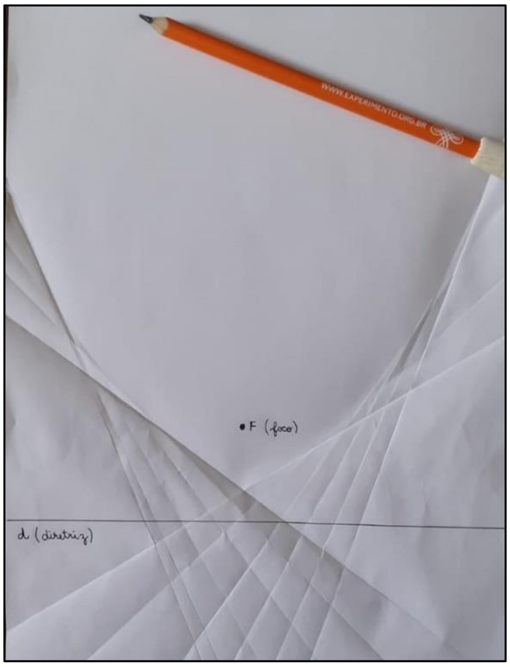

Figure 2. Parable with paper folds, manual construction

We also bring the possibility of transposing this construction carried out with pencil and paper in a physical way to the GeoGebra dynamic geometry environment, as in Figure 3:

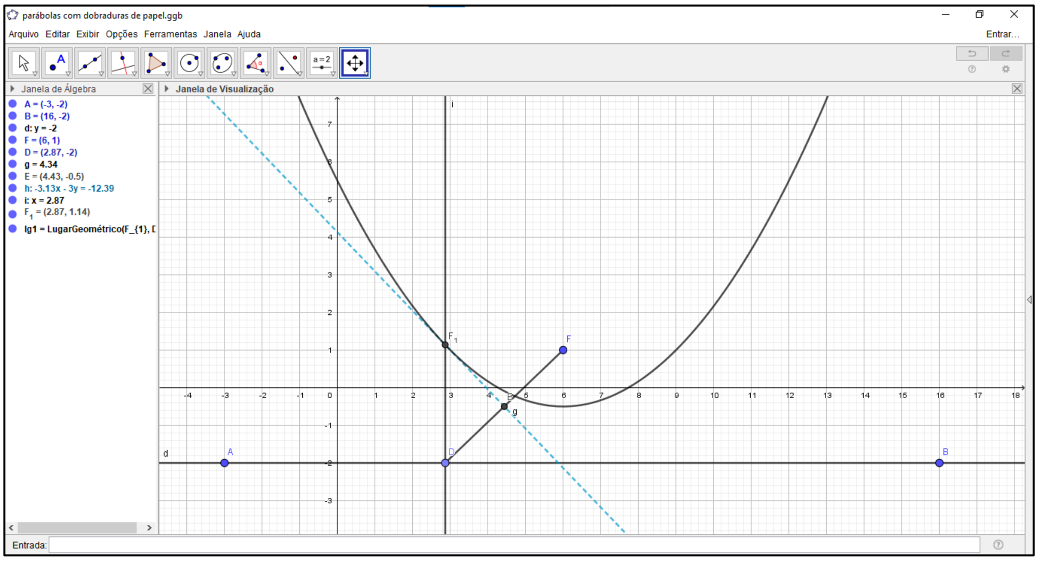

Figure 3. Structure of the parabola with folds in the GeoGebra environment

The line h in blue, shown in Figure 3, represents a tangent line to the parabola with focus F and directrix d. By enabling the "display trace" function, we have the various tangent lines that can be traced and that correspond to the parabola locus, as illustrated in Figure 4:

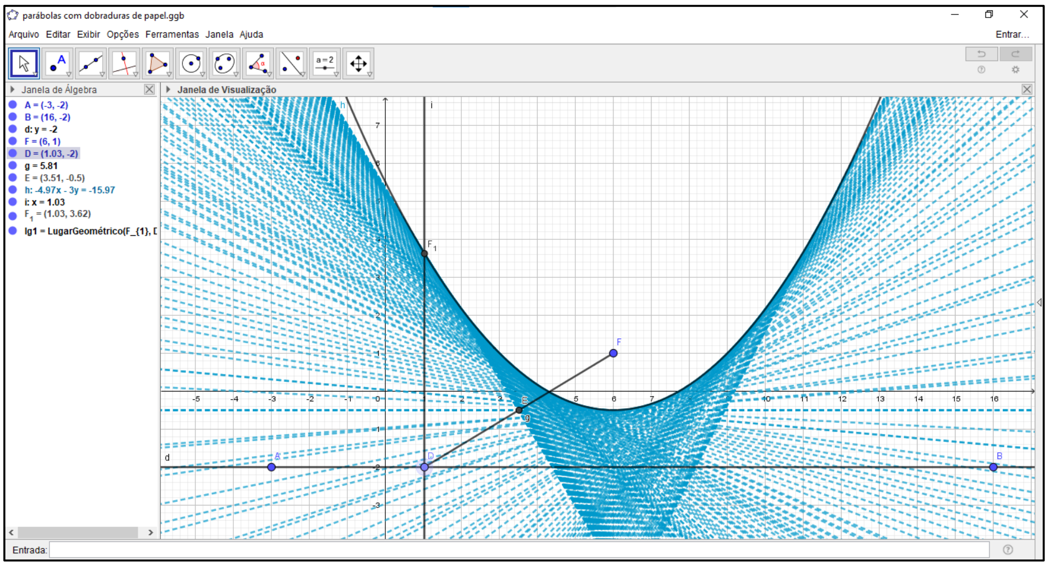

Figure 4. Tangent lines to the parabola of focus F and directrix d

The tangent lines in blue correspond to paper folds and can be built in a sequence of simple steps in GeoGebra, both in the software and in the smartphone app version:

1. Construct a line d that will be the directrix of the parabola and a point F outside the line d, which will correspond to the focus of the parabola;
2. Build a sliding point D on the directrix d, using the "point in object" tool;
3. Draw a line segment connecting point D to point F;
4. With the "midpoint" tool, construct the midpoint of the segment $\overline{DF}$;
5. Draw a line perpendicular to the segment $\overline{DF}$, passing through the constructed midpoint;

6. Draw another line perpendicular to the directrix line d, passing through the sliding point D on it;
7. With the “intersection between two objects” tool, construct the point of intersection between the two constructed perpendiculars;
8. With the “geometric place” tool, click on the sliding point D on the directrix line and the intersection point of the two perpendiculars;
9. A locus corresponding to the parabola of focus F and directrix d will appear.;
10. The perpendicular line passing through the midpoint of $\overline{DF}$ is tangent to the parabola. When clicking on this line with the right mouse button and "enable trail," move the sliding point D over the directrix, and the tangent lines to the parabola, corresponding to the paper folds, will be traced.

This construction is available on the geogebra.org community and can be accessed through the electronic address: https://www.geogebra.org/m/ujp8gymj. With the GeoGebra commands, the student can clearly understand each of the parabola elements and manipulate them, stimulating their geometric perception from the visualization.

- **Building the parabola with a ruler and compass**

Another possibility of constructing the parabola as a geometric place is through geometric drawing techniques with ruler and compass instruments.

When considering a plane β, a directrix line d contained in β and a point F inside the plane but not belonging to d, we have that the parabola is the locus of the points in the plane equidistant from F and d, as represented in GeoGebra and illustrated in Figure 5:

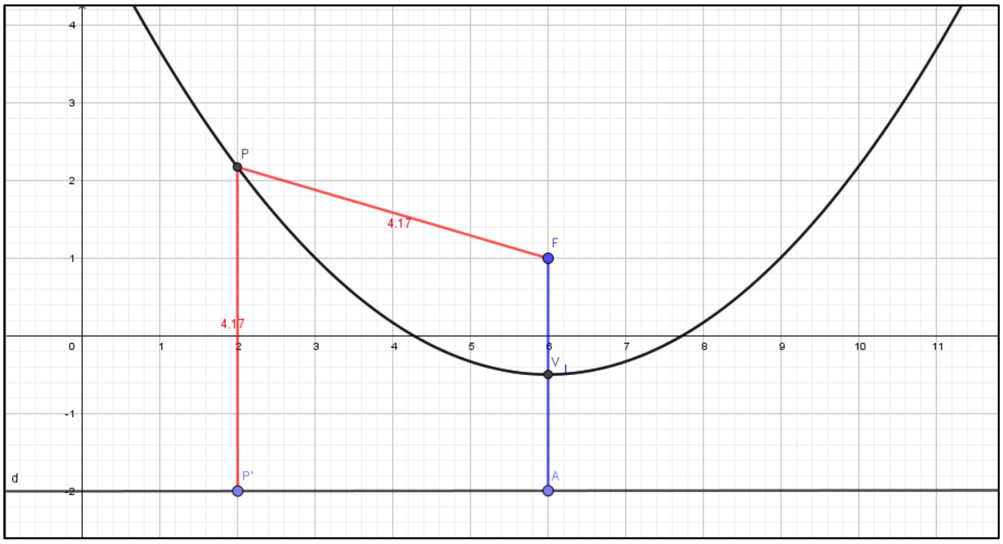

Figure 5. Definition of the parabola as a geometric locus

Note that for the described curve to be a parabola, the distance between PF and PP' must have the same measure (which appears in the construction with the measure 4.17, but when we move the point P', this value changes, keeping the condition $PF = PP'$). Another observation is that as F approaches A, the parabola tends to a straight line (which, in fact, occurs when $F = A$). Thus, we can start the sketch of this construction based on the sequence of steps described in Kilhian (2011):

1. Draw a directrix line d and any point F outside d, which will be the focus of the parabola;
2. Draw a line perpendicular to d passing through F. The point of intersection of the perpendicular line and the directrix d is our point A (as in Figure 4);
3. Get the midpoint of the segment $\overline{FA}$ as the point V, the vertex of the parabola;
4. Draw another line r parallel to d, at a distance h_1;
5. Draw as many lines parallel to d as you wish, considering the distances to be h_2, h_3, ..., h_n. Measure these distances with the opening of the compass itself;
6. Place the dry point of the compass at F and the aperture forming a radius equal to h_1. From this, describe an arc intercepting r_1 at points P_1 and P_1';
7. Then, opening the compass with a radius equal to h_2, draw another arc intercepting r_2 at P_2 and P_2', and so on, until the points P_n and P_n';
8. The parabola will be the curve that passes through V and the pairs of points P_1 and P_1', P_2 and P_2', and P_3 and P_3' up to P_n and P_n'.

An outline of this model can be presented in Figures 6, 7, and 8:

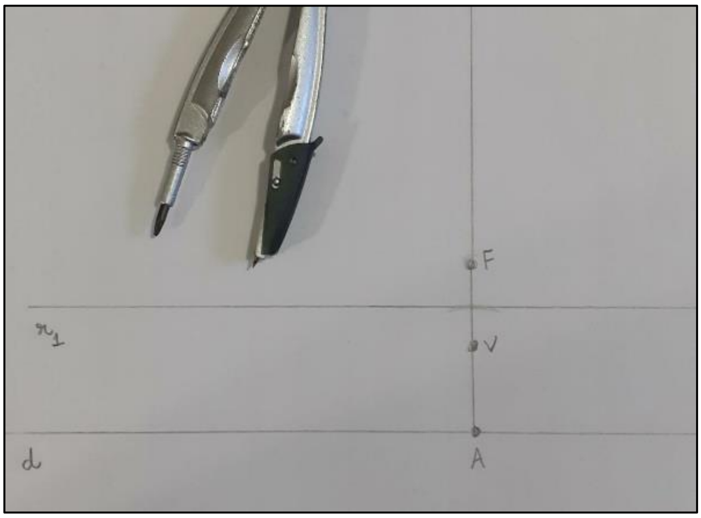

Figure 6. Steps 1 to 3

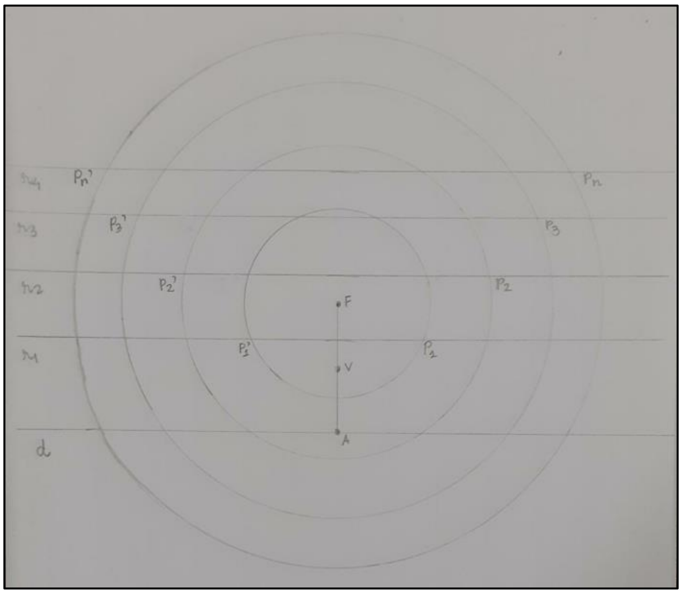

Figure 7. Steps 4 to 7

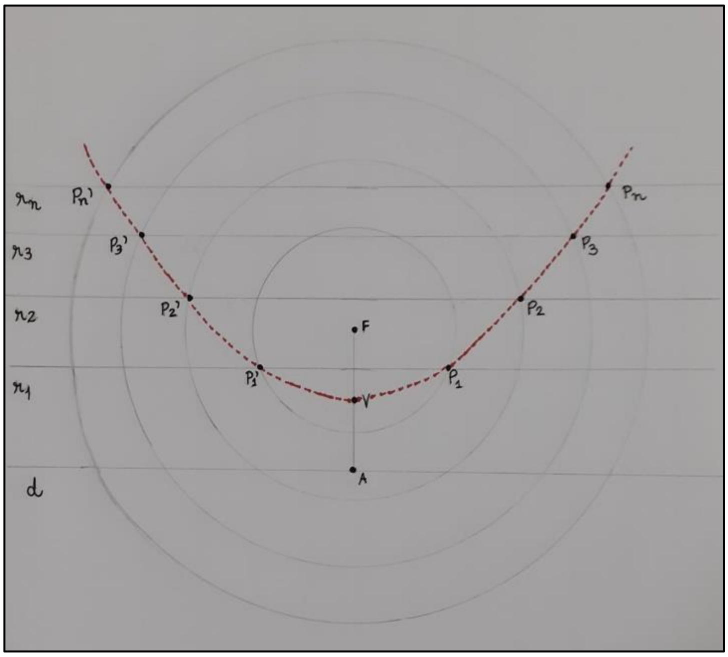

Figure 8. Step 8

Note that the curve passing through the points V, P_n, and P_n' forms the indicated parabola. We can also perform this construction in GeoGebra, as illustrated in Figure 9:

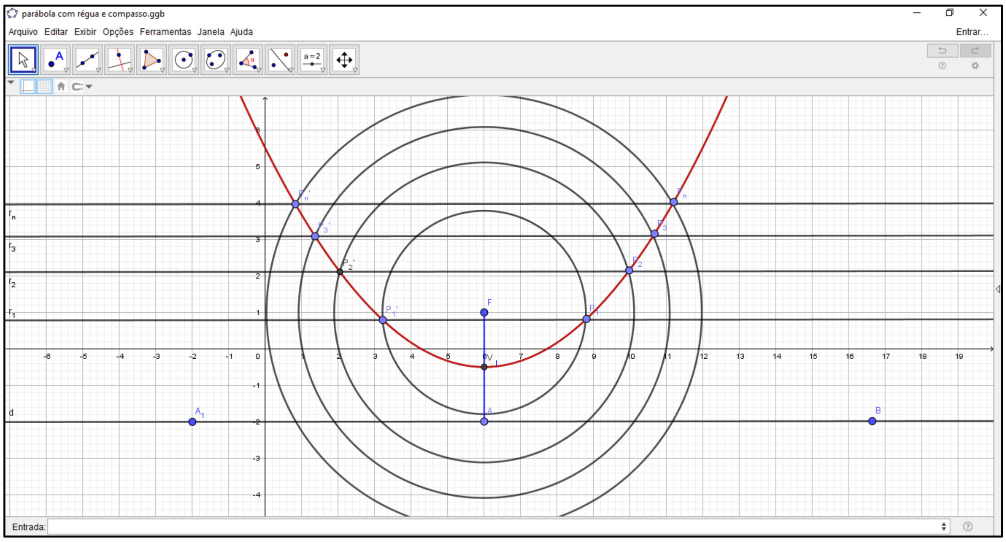

Figure 9. Parabola with ruler and compass illustrated in GeoGebra

It is possible to notice the symmetry of the parabola when we draw a line that passes through the segment $\overline{FA}$ and even discuss it in the classroom, comparing it with other models of construction of the parabola. This construction is available for use at: https://www.geogebra.org/m/kc7axkh6.

- **The parabola from the centers of circles**

A less common, but no less important, definition for the parable is presented by Souza Júnior, and Cardoso (2003) in *Revista do Professor de Matemática* (RPM edition 68), in which he shows how we can consider the parable from a different perspective than commonly found in textbooks, using circles.

The usual definition of a parabola is: The parabola with focus F and straight directrix d is the locus of the points equidistant from the focus and the directrix. If P is a point on the parabola, the perpendicular to d through P cuts d at T, such that $PT = PF$. Therefore, $C(P, PF2)$ is tangent to d at T (Souza Jr. & Cardoso, 2003). This construction can be reversed so that we arrive at another definition of parabola, equivalent to the usual one: The parabola is the locus of the centers of the circles that contain the point F and touch the line d.

So, we can infer that: given a focus F and a directrix d, let us consider all circles that pass-through F and are tangent to d. The set of centers C of all these circles corresponds to a parabola with focus F and directrix d. Such a definition can be seen in Figure 10:

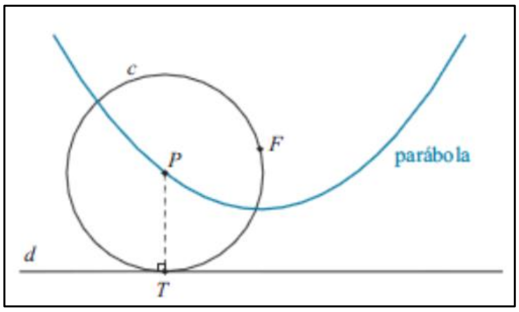

Figure 10. Parabola from circles - RPM 68

GeoGebra, as suggested by Souza Júnior and Cardoso (2003), brings a way to understand the definition indicated in Figure 9 more dynamically, as exemplified in Figures 11 and 12:

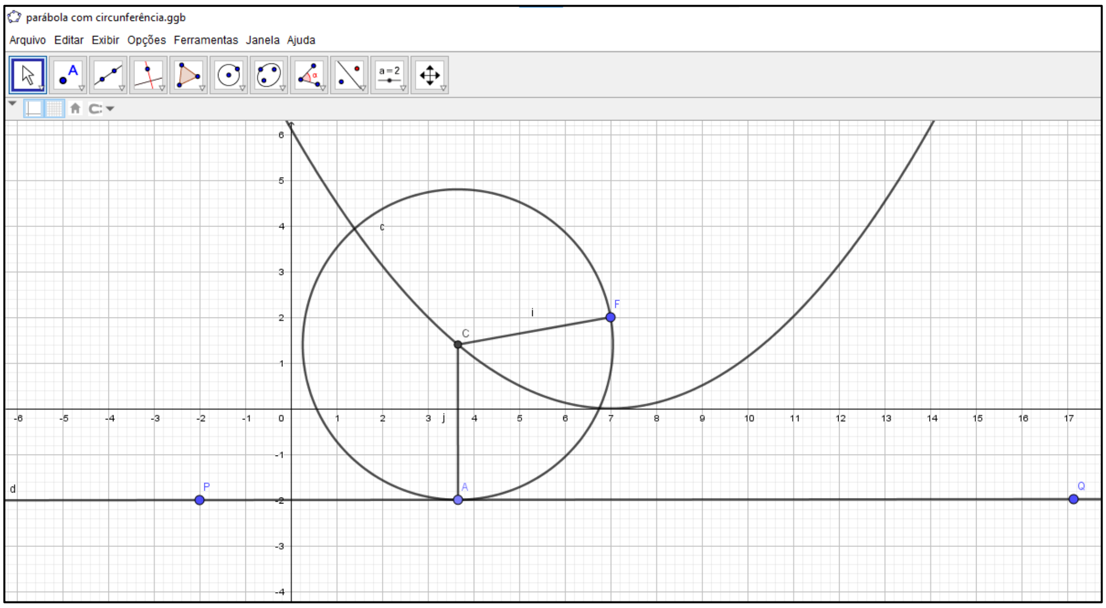

Figure 11. Parabola from circles

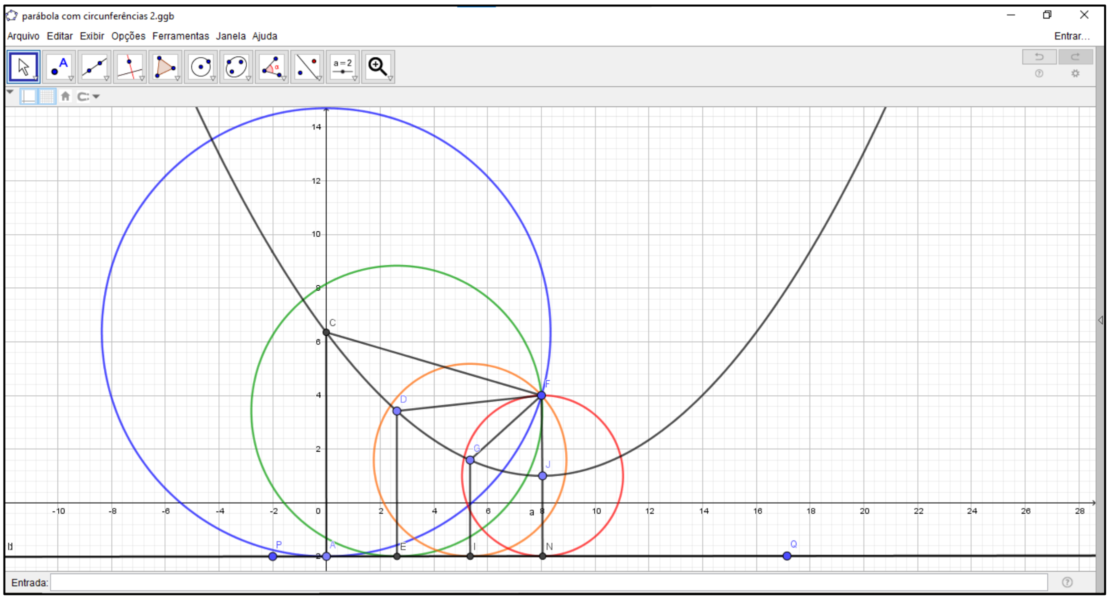

Figure 12. Parabola from circles, replication of circles

It is worth mentioning that this approach is unusual. But it also uses the distance between two points in Analytical Geometry to prove that the circle's radius corresponds to the measure of the distance from the center of this circle to the directrix, meeting the mathematical definition of a parabola.

Part of the protocol of this construction is similar to the construction presented in the previous subsection, with the difference in seeing the circumferences and their radii. This build is available on the geogebra.org community at https://www.geogebra.org/m/sdkgzavy and is ready for classroom use.

- **The parable in the context of functions**

Another possibility of working with the parabola, exploring the context of functions, is to present the equation of the parabola with vertex $V(h,k)$ outside the origin and seek a comparison with the quadratic function of the same vertex $V(h,k)$. For this, we build in GeoGebra the equation $(eq1)$ o and the quadratic function (f) through the sliders h and k (coordinates of the parabola vertex), a and p (parameters that define the opening of the concavity of the parabola and its position – up or down), where $a = \frac{1}{2p}$. Let's look at this construction in Figure 13:

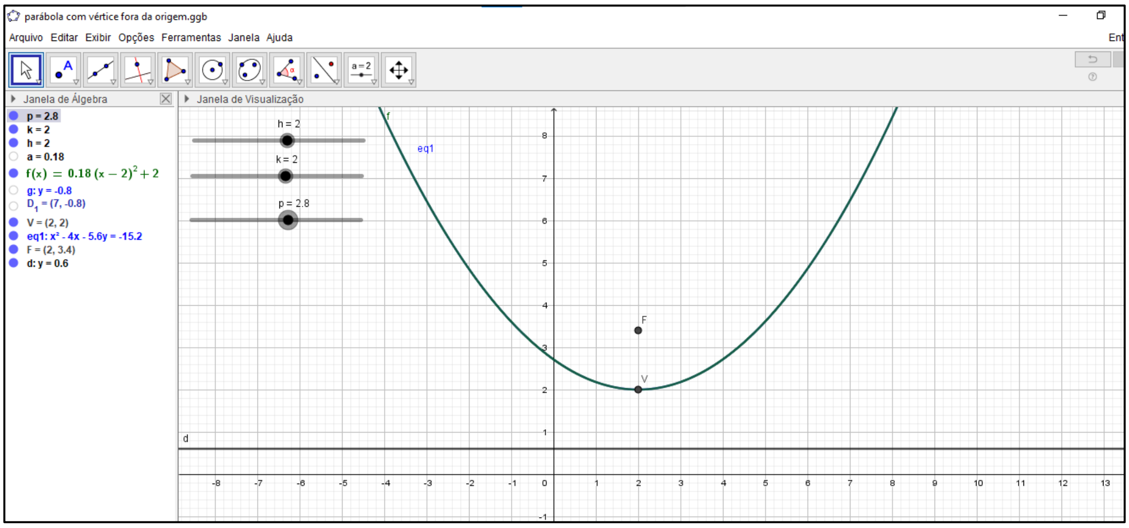

Figure 13. Comparison between the parabola equation and the canonical form of the quadratic function

Note that the graph $eq1$ represents a parabola with vertex $V(h, k)$ outside the origin and focus F and that there is a relationship between the expressions $f(x)$ and $eq1$ based on the parameters h, k, and p. By moving the h and k sliders, we define where the vertex of the parabola will be. In the slider p, we have a movement that reflects the change in the concavity of the parabola (opening or closing, changing its position up or down), in addition, to directly influencing the distance between the focus and the directrix (the greater the value of p, the greater the distance between the focus and the directrix of the parabola).

Note that this way the student can more clearly observe the relationship between the expressions $f(x) = a(x - x_v)^2 + y_v$, with $V(x_v, y_v) = (h, k)$, which corresponds to the canonical form of the quadratic function and $(x - h)^2 = 2p(y - k)$, as an expression that represents the equation of the parabola with vertex outside the origin (it is worth mentioning that the point (h, k) can assume the coordinates $(0,0)$ in the construction).
It is important to remember that this construction can be modified to obtain a parabola with a symmetry axis parallel to the x-axis from an expression of the type $(y - k)^2 = 2p(x - h)$ at $eq1$. This construction is available on the geogebra.org community via the link https://www.geogebra.org/m/dtpazcyg for classroom use.

- **General and reduced equation of the parabola**

This didactic proposal is aimed at teaching the parable at a higher level, as in undergraduate courses in Mathematics, for example. We propose its use for Analytical Geometry classes in undergraduate courses in Mathematics or related areas that study the topic. As it is a complex construction to be built in the pedagogical time of a class, we propose an activity model to work with the observation and manipulation of its terms. This construction is available for classroom use on the GeoGebra.org community at: https://www.geogebra.org/m/j59xaqux.

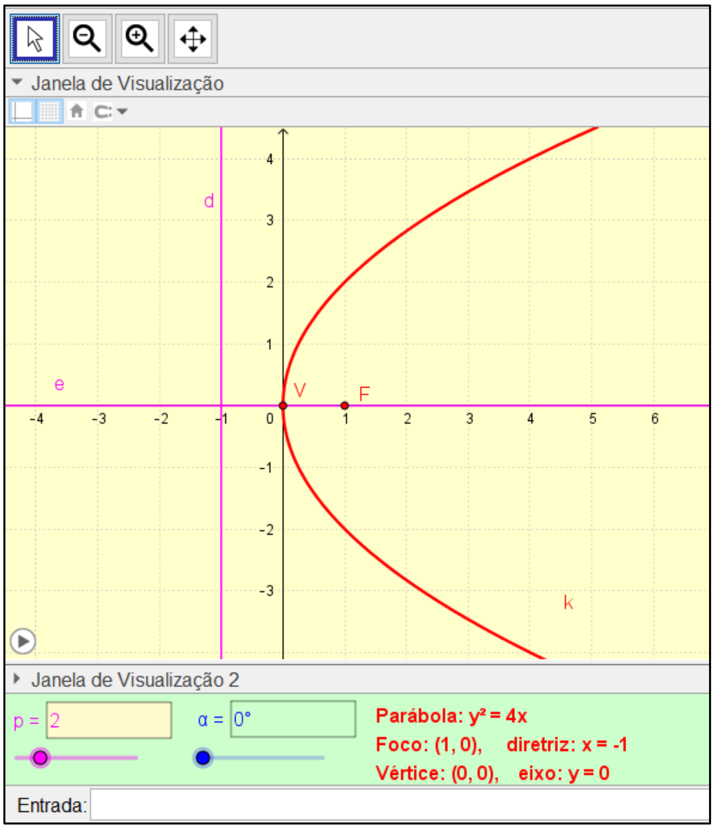

Figure 14. Construction for the general and reduced parabola equations

The teacher can request as an activity proposal that the student manipulates the vertex V of the parabola to move it and modify the angle α to rotate it, observing different aspects such as:

a) What new terms appear in the general equation if we move it without rotating it? How, then, is the equation reduced?

b) If we rotate it through an angle α ≠ 90º, what new term appears?

This activity deals with the different possibilities of rotation and translation of the parabola, in which we can observe the movement of the position of its axis of symmetry when manipulating the angle α, its translation from the movement of the coordinates of the vertex V, the opening of the concavity to the moving the parameter p, as well as the consequent change in its equation, which allows for a broader discussion on the topic.

By manipulating the vertex V of the parabola, students can shift it in the plane and notice that the equation of the parabola in its reduced form has the form $y^2 = 2px$ or $x^2 = 2py$ and it will appear in the construction only in the case where let $V(0,0)$ and α be an angle belonging to the axes (0°, 90°, 180°, 270° or 360°).

Based on their visual perception, students may notice these two cases:

(i) If $V \neq (0,0)$, but α is an angle belonging to the axes: we will have an explicit equation of the type $y = ax^2 + bx + c$ (for α = 90° or α = 270°), or the type $x = ay^2 + by + c$, for α = 90° or α = 270°);

(ii) If $V \neq (0,0)$ and α is an angle different from $0°, 90°, 180°, 270°$ or $360°$: we will have a general equation of the type $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, that is, an equation in its complete general form.

As pointed out by Lagrange et al. (2003) and Ramirez, Osorio, and Goycochea (2021), GeoGebra integrates algebra and calculus tools, allowing you to approach geometry in a dynamic environment. Thus, from this construction and its different possibilities of manipulation and visualization, the student can understand or even represent families of parables when experiencing this construction.

Different ways of visualizing this initial manipulation can be sketched. We seek to illustrate some of these possibilities in Figure 15:

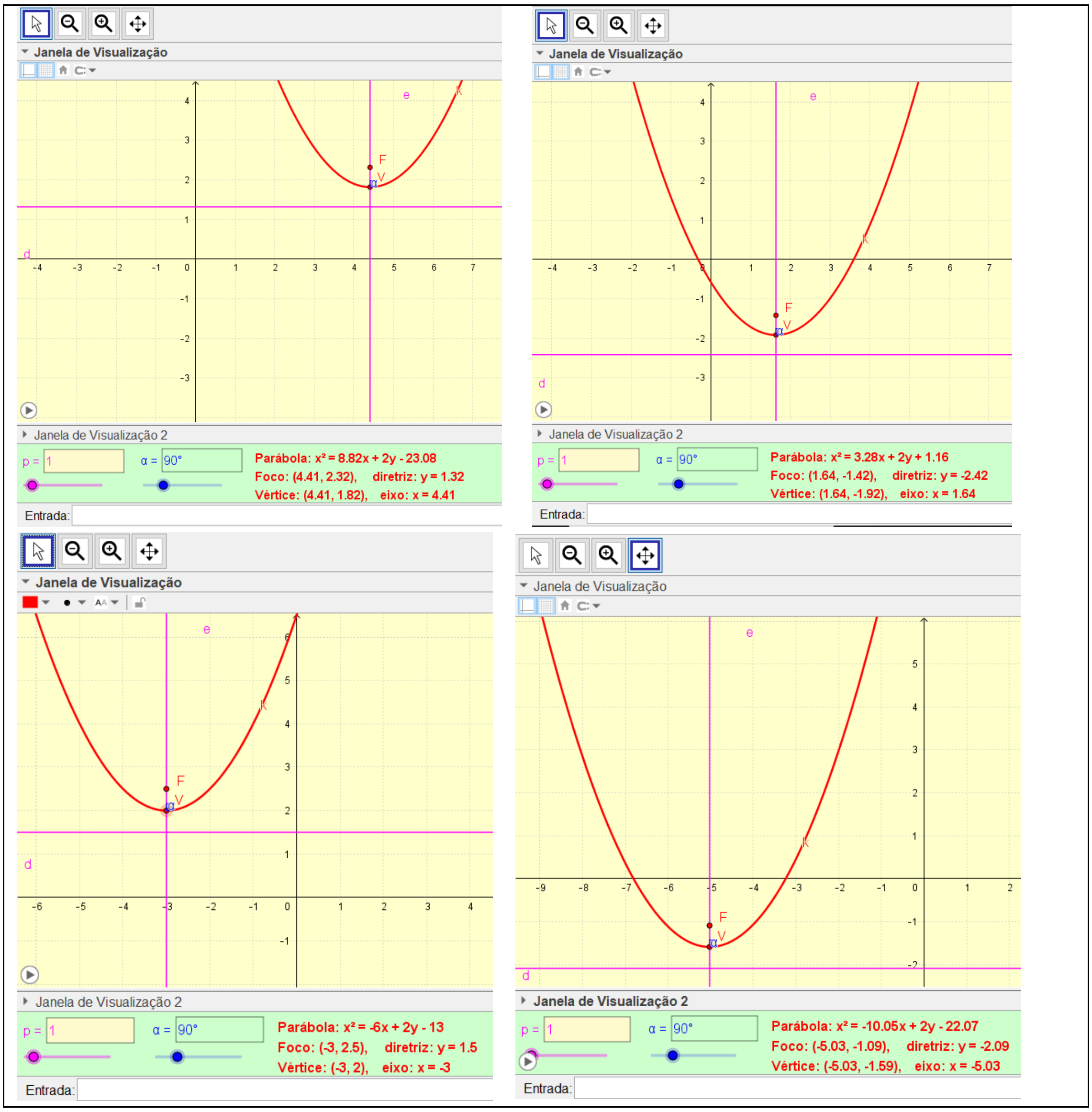

Figure 15. Example of manipulation of the V vertex in GeoGebra

Students may realize that by moving the parabola without rotating it, the equation will no longer be of the type $x^2 = 2py$, but rather an equation $x^2 = ax + by + c$, which in its reduced form can be written as $(x - x_0)^2 = 2p(y - y_0)$, given the fact that, under the conditions given at first, we have $p > 0$.

Another possibility for visual exploration of the parabola would be from an angle $\alpha \neq 90°$. This angle corresponds to the opening formed between the abscissa axis and the symmetry axis of the parabola. Here is an example of movements of the parameter p and the angle α, as outlined in Figure 16:

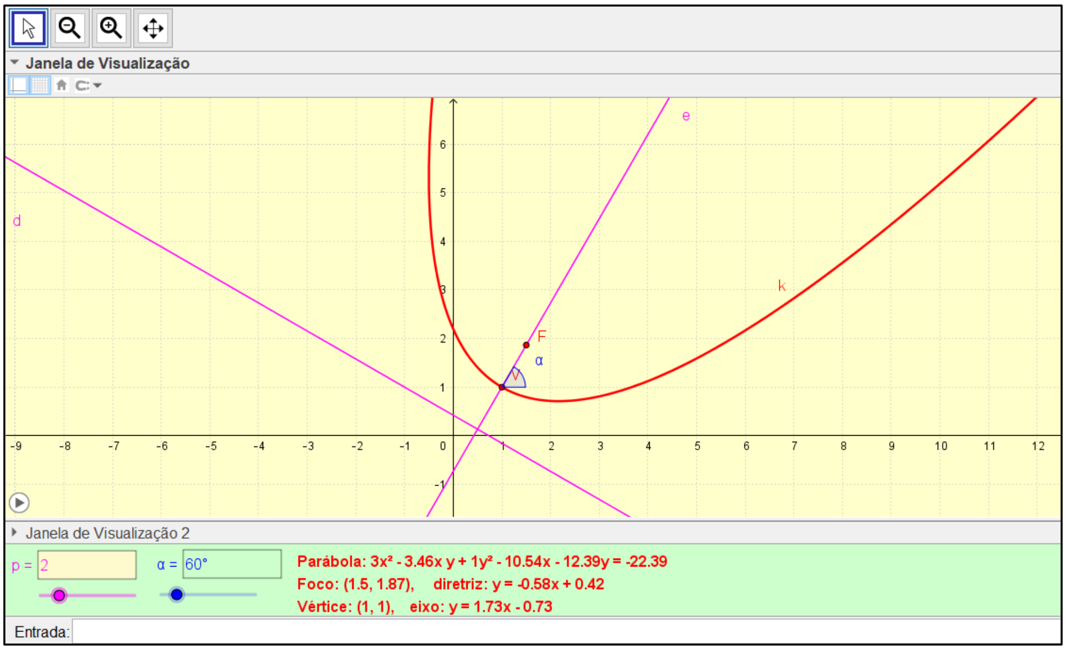

Figure 16. Parabola when $V \neq (0,0)$ and α do not belong to the coordinate axes

Therefore, the parabola equation has the general format $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, which students can discuss in the classroom, demonstrate, and justify.

The differential equations of the parabola and their possible variations to be explored with this construction, based on the coordinates of the vertex and the position of the axis of symmetry:

(i) Vertex at the origin and axis parallel to the coordinate axes;
(ii) Vertex at the origin and axis not parallel to the coordinate axes;
(iii) Vertex outside the origin and axis parallel to the coordinate axes;
(iv) Vertex outside the origin and axis not parallel to the coordinate axes.

An equation of the parabola in its general form, as mentioned by Venturi (2003), who calls a conic the "set of points in the plane whose Cartesian coordinates satisfy a 2nd-degree equation with two variables", of the type:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Being a complete equation when all its coefficients are not zero. In this case, from the construction, it is also possible to observe that the term Bxy, when $B \neq 0$, shows that the focal axis of the conic is oblique to the Cartesian axes.

Finally, we hope that the proposed activities can be used as complementary material to the teacher's methodology in their Mathematics classes, as well as by students in initial training who seek to expand their range of knowledge, exploring new possibilities for working on this subject.As a future perspective, it is intended to develop this research more broadly in the master’s course, seeking to improve observations and ways of working with parables, contributing to teacher training.

CONCLUSIONS

Parabolas, in addition to representing the quadratic (or 2nd degree) functions geometrically, are also the result of the study of conics in Analytical Geometry. The theoretical framework indicates that this subject is approached in a laconic way, in an analytical/algebraic view, not exploring the geometric characteristics and possibilities of the parabola nor relating the focus elements, directrix, vertex, and axis of symmetry to the study of quadratic functions, as a way of making a parallel between these two subjects.

Starting from this premise, we seek to explore the parable and its characteristics in this work, aiming at an understanding that relates to different ways of building the parable and understanding it. The constructions presented can be explored in High School and Higher Education classrooms.The limitations of this study are that this research is partial, demanding its continuity for application and data collection in the classroom to verify the feasibility of the didactic proposal. We also did not find in the textbooks used in Brazilian schools the study of this topic with GeoGebra or any other technological resource.

In addition, we also brought the use of technology in the exploration of its elements using the GeoGebra dynamic geometry software, from some possibilities of geometric construction of the parabola, to subsidize the work of the mathematics teacher and offer a reading that helps the student in the understanding of this subject, through geometric visualization.

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AUTHOR CONTRIBUTIONS STATEMENT

FRV designed the instruments. RTS and MJA collects data and analyzes the data that has been obtained. Furthermore, the authors co-authored this manuscript.

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