Solution of Klein-Gordon Equation in F(R) Theory of Gravity

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ABSTRACT

The $f(R)$ theory, as a modification of the general relativity theory, is frequently employed as an alternative theory of gravity and offers a promising avenue for addressing the challenges of formulating a quantum gravity theory. In this study, by applying the separation method of time, radial and angular variables, we derived the general solution of the Klein-Gordon equation in a curved space-time using modified Schwarzschild metric. We modified Ricci scalar R form in Einstein's action principle as a general function of Ricci scalar $f(R)$ and formulated the general Schwarzschild metric. The solution of the time function was analytically obtained in exponential form, and the solution of the angular function in terms of Legendre polynomial depends on azimuthal and magnetic quantum numbers. The radial function in terms of a non-linear second-order differential equation was solved by a numerical method using Python. The solutions described the gravitational effect for a light particle on the area gravitationally has a strong interaction, represented by a spherically symmetric metric. For small r (in Schwarzschild radius), the results analytically show that the gravitational effect in this region is massive. It follows that even light would be drawn into a black hole and unable to escape. For further research, it is expected to extend the Klein-Gordon equation in relativistic quantum mechanics to modified general relativity theory. This theory offers a different way of looking at the effects of gravity in quantum field theory.

INTRODUCTION

Black holes are currently a topic of great interest and remain a significant puzzle for scientists (Renner & Wang, 2021; Polchinski, 2017). Researchers strive to comprehensively understand the phenomena and processes that occur in black holes, including modifying Einstein's field equations in general relativity theory (Straight et al., 2020; Shankaranarayanan & Johnson, 2022). General relativity theory can explain astronomical phenomena, focusing on the structure of massive objects like neutron Stars, black holes, quasars, and the universe's expansion (Yagi & Stein, 2016; Barausse, 2019).

The discrepancy between observational and calculation results generated ideas to alter Einstein's general relativity theory. The first hypothesis changed the right side of the Einstein field equation by suggesting the existence of dark matter. The second idea modified the left side of the Einstein field equation by assuming only matter, like dark matter (Yadav & Verma, 2019) and dark energy (Odintsov & Oikonomou, 2019).

Two variational ideas that could be utilized to develop the modified general relativity theory are Formalism based on the Palatini variation and standard metric variation (Ferraris et al., 1982).

These principles are built on the field equation with the Lagrangian linear in R

(Buchdahl, 1970). Brans and Dicke finished the extension of the general relativity theory with the development of the scalar-tensor theory of gravity. One specific example of how gravitational interactions in general relativity theory are connected to the scalar field and the tensor field is the scalar-tensor theory. (Brans & Dicke, 1961).

The $f(R)$ theory, which is a part of the metric or Palatini formalism, and scalar-tensor theory, introduced the basic ideas of gravity theory (Capozziello & Laurentis, 2011; S. Capozziello et al., 2010). Starting with the spherical symmetry solutions in $f(R)$ theory, one could use the Noether symmetry approach to solve the axial symmetry problem. (Capozziello et al., 2010). One of the simplest modifications to gravity theory is the $f(R)$ theory, which generalizes the scalar Ricci of the Hilbert-Einstein equation to the function $f(R)$ of R. (Capozziello et al., 2010).

To construct general relativity theory using a semi-classical framework, the formulation of modified general relativity theory was proposed. The most successful method was determined using the $f(R)$ theory. (Capozziello et al., 2010). It has become a new framework for explaining the interaction of gravity (Faraoni & Capozziello, 2011).

The Klein-Gordon equation is beneficial for describing particles in the relativistic quantum mechanics (Bussey, 2022). This equation appears when the effect of relativity is calculated ($v \approx c$). The Klein-Gordon in linear form is a second-order partial differential equation. It is a relativistic wave function that represents the dynamics of elementary particles on a relativistic scale (Joseph, 2020). In this research, we also prove how the solution of the Klein-Gordon equation in modified gravity $f(R)$ theory is consistent with the non-relativistic limit.

A simple function represents the solution of the Klein-Gordon equation in Schwarzschild space-time analytically solved for region $0 \leq r \leq \infty$ (Elizalde, 1988; Qin, 2012). In contrast, the Klein-Gordon equation for time-dependent solved using an asymptotic method for certain angular momentum conditions and proved it for the Schwarzschild radius (Rowan & Stephenson, 1976).

Some modifications of general relativity such as $f(G)$, $f(R)$, and gravity $f(R, G)$ as a gravitational modification, were constructed to explain unsolved phenomena like dark matter, inflation, etc. (Nojiri & Odintsov, 2007) and (Multamäki & Vilja, 2006). Capozziello et al. reviewed and introduced the fundamental principles of the theory of gravity, more specifically to the scalar-tensor theory and the $f(R)$ theory (Salvatore Capozziello & Francaviglia, 2008; Capozziello et al., 2012).

The confluent Heun functions provide the angular and radial parts of the Klein-Gordon equation solutions. This study clarified how a charged, rotating black hole's gravitational field (Kerr-Newman space-time) affects a charged, massive scalar field (Vieira et al., 2014). Numerical methods are needed to solve the KG equation in curved space-time (Lehn et al., 2018a; Griffith, 2004). Earlier research that sought exact solutions, which either involved approximative expansions or simplifying assumptions to obtain asymptotic solutions, could have been more conclusive at best. (Lehn et al., 2018a). The Klein-Gordon equation is used for a massless scalar field contained in a Casimir cavity and moves in an equatorial orbit (geodesic) (Sorge, 2014).

The solution of Klein-Gordon equations has also been worked out for other black hole models, which succeeded in numerical solution as a periodic function of a black hole (Pourhassan, 2016). The solution of the relativistic Klein-Gordon equation in curved space-time with a massive field was obtained numerically and then compared to the non-relativistic Coulomb field solution directly through the interference theory (Lehn et al., 2018b). The solutions explain how the gravitational effect work. The spherically symmetric metric represents it.

From the previous research, we expand the solution of Klein-Gordon in curved space-time in general relativity theory by (Lehn et al., 2018a) to the modified gravity theory using the $f(R)$ theory. Some unsolved phenomena in general relativity, such as dark matter and inflation, encourage me to do it as an alternative theory of gravity. The investigation of a gravitational effect in $f(R)$ theory solving the Klein-Gordon equation has never been studied before.

METHODS

The solutions of the Klein-Gordon equations in several gravitational fields and their consequences are fundamental to discuss. It is important to note that, in principle, the physics of these things may be understood by looking at how scalar fields behave in black hole backgrounds. The Klein-Gordon equation must thus be solved for both natural and complex areas, and associated phenomena like the radiation of scalar particles must be investigated.

In this research, we generalize the Klein-Gordon equation in curved space-time for a light particle on the area that gravitationally has a strong interaction. A spherically symmetric metric represents it. First, we extended Einstein's field equation in general relativity to modified general relativity through $f(R)$ theory. Second, the metric for a static spherical solution in Schwarzschild space-time was constructed using $the\ f(R)$ theory of general relativity. It is known as modified Schwarzschild space-time. From Einstein's action principle as a general function of Ricci scalar $f(R)$, the general Schwarzschild metric was formulated. Third, we solved the Klein-Gordon equation by substituting the modified Schwarzschild metric by using the method of separation of variables. Lastly, the general solution of Klein-Gordon was derived in exact calculation by separating time, radial, and angular variables, and the numerical solution of the radial equation is also presented. The radial function in terms of a non-linear

second-order differential equation was solved by a numerical method using Python.

RESULTS AND DISCUSSION

$f(R)$ Theory of General Relativity

$f(R)$ theory of general relativity is one modification of gravity theory first proposed by Hans Adolph Buchdahl in 1970. Generalization of Einstein-Hilbert's action to become a general function of as follows

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + L^{(m)} \qquad (1)$$

where $L^{(m)}$ is the matter Lagrangian, and $f(R)$ is a function of the Ricci scalar. We obtained the Einstein field equation in general relativity theory by setting $f(R) = R$.

By applying the variational principle of the action in Equation (1), we found the generalization of the fields equation in the $f(R)$ function (Sotiriou & Faraoni, 2010)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$
$$= \frac{1}{f'(R)} \left[\nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \nabla_\mu \nabla^\mu f'(R) \right. \qquad (2)$$
$$\left. + \frac{\{f(R) - f'(R)R\}}{2} g_{\mu\nu} \right] + T_{\mu\nu}$$

or in a different form Equation (2) is written as

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2} g_{\mu\nu} - f'(R);\mu\nu$$
$$+ g_{\mu\nu} \nabla_\mu \nabla^\mu f'(R) \qquad (3)$$
$$= \kappa T_{\mu\nu}$$

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2} g_{\mu\nu} - H_{\mu\nu} = \kappa T_{\mu\nu} \qquad (4)$$

where $H_{\mu\nu} = f'(R);\mu\nu - g_{\mu\nu} \nabla_\mu \nabla^\mu f'(R)$, the trace of Equation (3) and (4)

$$3\nabla_\mu \nabla^\mu f'(R) + f'(R)R - 2f(R) = \kappa T$$
$$f'(R)R - 2f(R) - H = \kappa T. \qquad (5)$$

Schwarzschild Metric on $f(R)$ Theory

We calculate the modified Einstein field equation using $f(R)$ theory for the Schwarzschild metric with a static object of mass m and spherical symmetry. The metric of spherical symmetry can be written

$$ds^2 = -a(r)dt^2 + b(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad (6)$$

with Ricci curvature scalar R can be simplified following the Bernoulli equation

$$b'(r) + h(r)b(r) + l(r)b^2(r) = 0 \quad (7)$$

the value of $h(r)$ dan $l(r)$ are linear constants, and the square of $b(r)$

$$h(r) = \frac{r^2 a'^2(r) - 4a^2(r) - 2ra(r)[2a'(r) + ra''(r)]}{ra(r)[4a(r) + ra'(r)]} \quad (8)$$

and

$$l(r) = \frac{2a(r)}{r}\left[\frac{2 + r^2 R(r)}{4a(r) + ra'(r)}\right] \quad (9)$$

solution of Equation (8)

$$b(r) = \frac{exp\left(-\int dr\, h(r)\right)}{K + \int dr\, l(r)\, exp\left(-\int dr\, h(r)\right)} \quad (10)$$

exact solution of Equation (8) is obtained if $l(r) = 0$ with $R = -\frac{2}{r^2}$. For $b(r)$, return to Minkowski in $r = \infty$, $h(r)$ and $l(r)$ must be zero, so $b'(r) = 0$. The value of R is equal to r^{-n} at a great distance from the center, so the Bernoulli solution is

$$a(r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} + \frac{1}{r^2}\int dr\left(\int r^2 R(r)dr\right) \quad (11)$$

with k_1 and k_2 is constants.

The Einstein field equation is modified using the $f(R)$ theory

$$3\nabla_\mu\nabla^\mu f'(R) + f'(R)R - 2f(R) = \kappa T$$
$$f'(R)R - 2f(R) - H = \kappa T \quad (12)$$

where $H_{\mu\nu} = f'(R)_{;\mu\nu} - g_{\mu\nu}\nabla_\mu\nabla^\mu f'(R)$. The solution of Equation (11) looks like de Sitter-Schwarzschild that

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad (20)$$

$$a(r) = 1 - \frac{2m}{r} - \frac{\lambda}{3}r^2 \quad (13)$$

so the modified Schwarzshild metric was obtained

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Klein-Gordon Equation on Modified Schwarzschild Metric

The relativistic energy of free mass m is (Bjorken et al., 1966), (Romadani & Rani, 2020)

$$E^2\Psi = (p^2 c^2 + m^2 c^4)\Psi \quad (15)$$

in quantum mechanics, E and p are operators where E is expressed by $i\hbar\frac{\partial}{\partial t}$, and p is expressed by $-i\hbar\nabla$ and substitute to Equation (15) becomes

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2} = \left(-\hbar^2\nabla^2 c^2 + m^2 c^4\right) \quad (16)$$

a simple formulation of Equation (16) is written by

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2}\right)\Psi = 0 \quad (17)$$

which is known as the Klein-Gordon equation in the Minkowskian metric. The general equation of the Klein-Gordon equation in tensor form is

$$\left(\nabla_\mu\nabla^\mu + m^2\right)\Psi = 0 \quad (18)$$

with $\nabla_\mu = g_{\mu\nu}\nabla^\nu$. Equation (19) becomes

$$\left[\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\right) + m^2\right]\Psi = 0 \quad (19)$$

the covariant metric $g_{\mu\nu}$ and contravariant metric $g^{\mu\nu}$ in modified Schwarzschild is written by (Romadani, 2015)

$$g^{\mu\nu} = \begin{pmatrix} -\left(1 - \dfrac{2m}{r} - \dfrac{\lambda r^2}{3}\right)^{-1} & 0 & 0 & 0 \\ 0 & \left(1 - \dfrac{2m}{r} - \dfrac{\lambda r^2}{3}\right) & 0 & 0 \\ 0 & 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & 0 & \dfrac{1}{r^2 \sin^2\theta} \end{pmatrix}. \tag{21}$$

The Klein-Gordon equation on spherically symmetric metrics in Equation (19) can be derived becomes

$$\left\{\frac{1}{g_{tt}}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{g_{rr}}\frac{\partial}{\partial r}\right)\right.$$
$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \tag{22}$$
$$\left.+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + m^2\right\}\Psi = 0$$

where $\sqrt{g} = r^2 \sin\theta$.

To solve Equation (22), we defined

$$\Psi(t, r, \theta, \varphi) = T(t)R(r)Y(\theta, \varphi) \tag{23}$$

by using the separation of variables with substitute Equation (23) to (22), we found that

$$\frac{1}{T}\frac{\partial^2}{\partial t^2} + g_{tt}\left\{\frac{1}{r^2 R}\frac{\partial}{\partial r}\left(\frac{r^2}{g_{rr}}\frac{\partial R}{\partial r}\right)\right. \tag{24}$$
$$+ \frac{1}{r^2}\left[\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)\right.$$
$$\left.\left.+ \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] + m^2\right\}\Psi = 0$$

Solution for Time Function

From Equation (24), time variable t is written as

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -E^2 \tag{25}$$

the solution of Equation (26) satisfies the boundary limit, and the solution is

$$T(t) = Ae^{-iEt}. \tag{26}$$

Solution for Angular Function

The equation of the angular part from Equation (24) is expressed as

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\varphi^2} \tag{27}$$
$$= -l(l+1)Y\sin^2\theta$$

by using the method of separation variables $Y(\theta, \phi) = \Theta(\theta)\Phi(\varphi)$, it can be derived become

$$\frac{1}{\Theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)\right] + l(l \tag{28}$$
$$+ 1)\sin^2\theta = m^2$$

for variable θ with the solution is

$$\Theta(\theta) = BP_l^m(\cos\theta) \tag{29}$$

where P_l^m is associated Legendre function depends on azimuthal quantum number l and magnetic quantum number m by definition satisfy

$$P_l^m(x) = (1 - x^2)^{\frac{|m|}{2}}\left(\frac{d}{dx}\right)^{|m|}P_l(x) \tag{30}$$

and $P_l(x)$ is the Legendre polynomial satisfy the Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2 - 1)^l \tag{31}$$

the angular equation for variable φ is written by

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2 \tag{32}$$

with the solution

$$\Phi(\varphi) = Ce^{im\phi} + De^{-im\phi} \tag{33}$$

because the solution has to cover the latter by allowing m to run negative, we found that

$$\Phi(\varphi) = Ce^{im\varphi} \tag{34}$$

the constants factor in front absorbs that into Θ where $0 \le \varphi \le 2\pi$. Equation (35) has to require $\Theta(\varphi + 2\pi) = \Theta(\varphi)$ so m must be an integer $m = 0, \pm1, \pm2, \dots$. The solution for angular function $Y_l^m(\theta, \varphi)$ is formulated in Table 1.

Tabel 1: The spherical harmonics functions $Y_l^m(\theta, \varphi)$ (Griffith, 2004)

l	m	$Y_l^m(\theta, \varphi)$
0	0	$Y_0^0(\theta, \varphi) = \left(\dfrac{1}{4\pi}\right)^{1/2}$
	-1	$Y_1^{-1}(\theta, \varphi) = \left(\dfrac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{-i\varphi}$
1	0	$Y_1^0(\theta, \varphi) = \left(\dfrac{3}{4\pi}\right)^{1/2} \cos\theta$
	1	$Y_1^1(\theta, \varphi) = -\left(\dfrac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{+i\varphi}$
	-2	$Y_2^{-2}(\theta, \varphi) = \left(\dfrac{15}{32\pi}\right)^{1/2} \sin^2\theta\, e^{-2i\varphi}$
	-1	$Y_2^{-1}(\theta, \varphi) = \left(\dfrac{15}{8\pi}\right)^{1/2} \sin\theta \cos\theta\, e^{-i\varphi}$
2	0	$Y_2^0(\theta, \varphi) = \left(\dfrac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$
	1	$Y_2^1(\theta, \varphi) = -\left(\dfrac{15}{8\pi}\right)^{1/2} \sin\theta \cos\theta\, e^{i\varphi}$
	2	$Y_2^2(\theta, \varphi) = \left(\dfrac{15}{32\pi}\right)^{1/2} \sin^2\theta\, e^{2i\varphi}$

Solution for Radial Function

The radial equation from Equation (24) is written by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{g_{rr}}\frac{\partial R}{\partial r}\right) + \left[m^2 - \frac{E^2}{g_{tt}} - \frac{l(l+1)}{r^2}\right]R = 0 \quad (35)$$

we define that $R(r) = \sqrt{g_{rr}}\,\dfrac{U(r)}{r}$, so the radial equation reduces to

$$\frac{d^2U}{dr^2} + \left[\frac{g'_{rr}}{r\,g_{rr}} + \frac{1}{2}\frac{g''_{rr}}{g_{rr}} - \frac{3}{4}\left(\frac{g'_{rr}}{g_{rr}}\right)^2 \right.$$
$$+ \left(m^2 + E^2 g_{rr} \right. \quad (36)$$
$$\left.\left. - \frac{l(l+1)}{r^2}\right)g_{rr}\right]U = 0$$

where g_{rr} is modified Schwarzschild space-time in Equation (20).

Numerical Solutions for Radial Function

Here, we see the numerical solution of the radial equation in Equation (36). We assume $\lambda = m = E = 1$ to predict the solution of $R(r)$ for $l = 0$, $l = 1$, and $l = 2$. Using Python, we found the numerical solution for the radial function in Tabel 2.

The numerical results for the radial function in Table 2 have been plotted in Figure 1, and we can see that the increasing azimuthal quantum number l is followed by the increasing slope of $R(r)$ (Bjorken et al., 1966).

Table 2. The numerical results of the radial equation for $l = 0$, $l = 1$, and $l = 2$

r	$l = 0$		$l = 1$		$l = 2$	
	$U(r)$	$R(r)$	$U(r)$	$R(r)$	$U(r)$	$R(r)$
1.01	0.0170	0.0194	0.0265	0.0302	0.0562	0.0640
10.01	-0.0451	-0.0257	-0.1406	-0.0802	-0.3184	-0.1816
20.01	0.0376	0.0216	-0.0860	-0.0494	-1.6285	-0.9370
30.01	0.0919	0.0530	0.1160	0.0669	-0.9560	-0.5511
40.01	0.0966	0.0557	0.2653	0.1530	0.3192	0.1842
50.01	0.0707	0.0408	0.3343	0.1929	1.5291	0.8823

r	$l = 0$		$l = 1$		$l = 2$	
	$U(r)$	$R(r)$	$U(r)$	$R(r)$	$U(r)$	$R(r)$
60.01	0.0302	0.0174	0.3367	0.1943	2.4474	1.4124
70.01	-0.0149	-0.0086	0.2922	0.1686	3.0313	1.7496
80.01	-0.0587	-0.0339	0.2174	0.1255	3.3078	1.9093
90.01	-0.0980	-0.0566	0.1254	0.0724	3.3262	1.9200
100.01	-0.1314	-0.0759	0.0255	0.0147	3.1390	1.8120
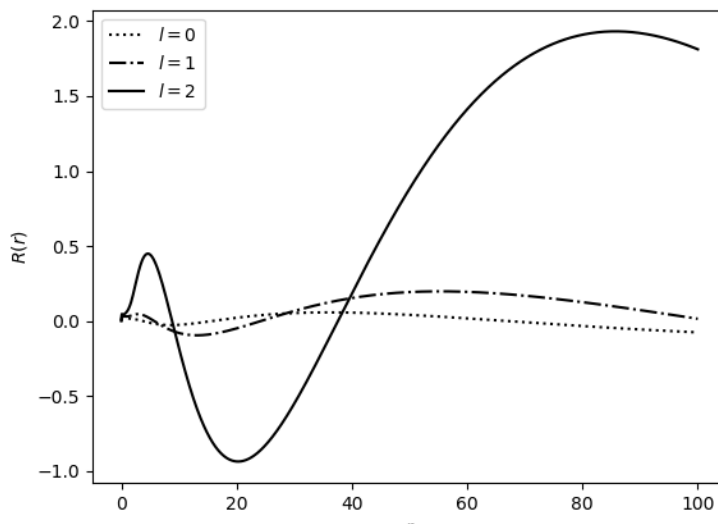
Figure 1. Comparison of radial functions $R(r)$ in $f(R)$ theory for $l = 0$, $l = 1$, and $l = 2$

Figure 1 shows the evolution of radial wave function as relativistic effects in $f(R)$ are increased with varying Schwrazschild radius. We numerically evaluate the expression of the radial wave function given by Equation (36). As the Schwarzschild radius increases (for small l), the wave function moves closer to the sphere (Griffith, 2004). For $\lambda \to 0$, the solution of the radial wave function in $f(R)$ reduced to the general relativity (Lehn et al., 2018a). However, the authors have previously demonstrated a gravitational effect similar to the Klein-Gordon equation without a λ parameter (standard general relativity). The radial wave solution behaves as a damped oscillator, and

in fact, it is a bound state solution that satisfies the boundary condition of being zero at infinity.

In Figure 2, the numerical results of radial solution are compared in both modified Schwarzschild in $f(R)$ theory and Schwarzschild metric for $l = 0$ and $l = 1$. In approximation, for $\lambda = 0$ in Equation (32), we get the standard radial function of the Klein-Gordon equation in Schwarzschild metric (Cruz-Dombriz et al., 2009). In other words, the red lines in Figure 2 equal the blue lines. It shows that the Klein-Gordon equation in $f(R)$ theory has a more general form rather than the standard Klein-Gordon equation.
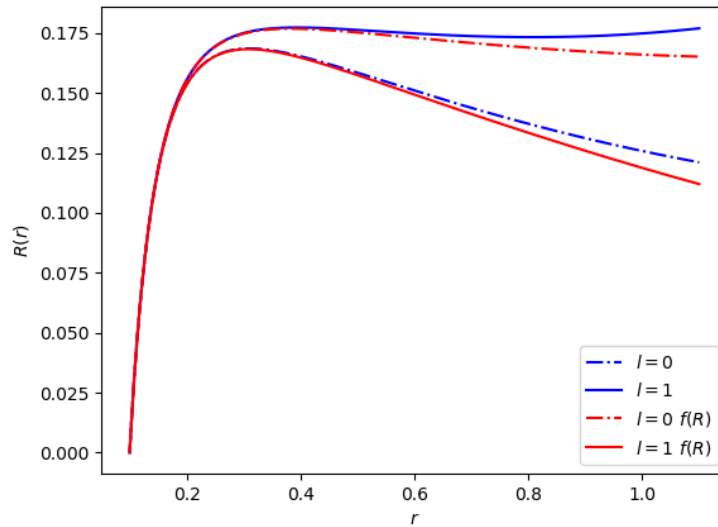
Figure 2. Comparison of radial functions $R(r)$ between $f(R)$ theory and standard theory for $l = 0$ and $l = 1$

In Figure 2, we compared the numerical results of radial wave function between $f(R)$ theory and general relativity theory for $l = 0$ and $l = 1$. We can say that the behavior of radial wave function in $f(R)$ theory is consistent with general relativity, especially in a small radius of r. We have shown that solutions of the Klein-Gordon equation in $f(R)$ theory can be computed with ordinary numerical methods and that the results are consistent with the non-relativistic limit. In general relativity theory for r fixed, we can see that increasing l also increases the value of $R(r)$. Still, in $f(R)$, the value of $R(r)$ decreases because for r greater, the radial part of Equation (36) is very dominant compared to the value of l.

The general solution of Klein-Gordon equation in $f(R)$ theory is written by

$$\Psi(t, r, \theta, \varphi) = \Psi_{lm}(t,r) = \kappa \, R(r) Y_l^m(\theta, \varphi) e^{-iEt} \tag{37}$$

where κ is a normalization constant. For $l = 0$ and $m = 0$ at $r = 50.01$ constant, we found that

$$\Psi_{00} = \kappa \left(\frac{0.00042}{\pi}\right)^{1/2} e^{iEt} \tag{38}$$

for $l = 1$ and $m = -1, 0, 1$ the solutions are

$$\Psi_{10} = \kappa \left(\frac{0.280}{\pi}\right)^{1/2} \cos\theta \, e^{iEt} \tag{39}$$

$$\Psi_{1,\mp 1} = \pm\kappa \left(\frac{0.0140}{\pi}\right)^{1/2} \sin\theta \, e^{i(Et \mp \varphi)} \tag{40}$$

and for $l = 2$ and $m = -2, -1, 0, 1, 2$

$$\Psi_{20} = \kappa \left(\frac{0.2433}{\pi}\right)^{1/2} (3\cos^2\theta - 1)e^{iEt} \tag{41}$$

$$\Psi_{2,\mp 1} = \pm\kappa \left(\frac{1.460}{\pi}\right)^{1/2} \sin\theta\cos\theta \, e^{i(Et \mp \varphi)} \tag{42}$$

$$\Psi_{2,\mp 2} = \kappa \left(\frac{0.3649}{\pi}\right)^{1/2} \sin^2\theta \, e^{i(Et \mp 2\varphi)} \tag{43}$$

Normalization

From the numerical solution, the radial function associated with the graph is $\approx r^{1/2} \exp(-r)$. According to Equation (35), the normalization of the general solution is

$$\int_{-\infty}^{\infty} d^3r \; \Psi_{lm}^\dagger(t,r)\Psi_{l'm'}(t,r) = 1 \tag{44}$$

$$\kappa^2 \int_{-\infty}^{\infty} d^3r \; r \exp(-2r) Y_l^{m\dagger}(\theta,\varphi) Y_{l''}^{m'}(\theta,\varphi) = 1 \tag{45}$$

where

$$\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} Y_l^{m\dagger}(\theta,\varphi) Y_{l'}^{m'}(\theta,\varphi) \, d\phi = \delta_{ll'}\delta_{mm'}$$

and the integral of the radial part is

$$\kappa^2 \int_0^\infty r^3 \exp(-2r) \, dr = 1$$

The normalization constant is $\kappa = \sqrt{\frac{8}{3}}$.

The numerical solution of the Klein-Gordon equation in this research is to build a mathematical formulation to solve the relation of gravitational effect in vacuum energy of parallel plates called by Casimir effect (Bezerra et al., 2014; Vieira et al., 2014; Bezerra et al., 2017). This research contributed to explaining the gravitational effect of the Klein-Gordon equation in curved space-time (Lehn et al., 2018a).

CONCLUSION AND SUGGESTION

We have derived a general solution of the Klein-Gordon equation in curved space-time, i.e., modified Schwarzschild metric whose solutions function as a complex variable. Modified Schwarzschild metric has been found by extending Einstein's field equation using the $f(R)$ theory. Analytically, we obtained the time-dependent Klein-Gordon and time-independent solutions shown by radial and angular functions. Radial function solution as a non-linear differential equation form was numerically solved using Python. The answer to the angular part depends on magnetic quantum number m and azimuthal quantum number l. The quantum numbers specify the properties of the atomic orbitals and the electrons in those orbitals, representing the solution of Klein-Gordon. In future work, we will continue the formulation of this research to solve the vacuum energy of the Casimir effect in curved space-time and solve the Dirac equation to explain fermion particles because Klein-Gordon is limited for Boson particles. Using all the obtained calculation results of $f(R)$ theory, the collaboration of modified general relativity theory to the relativistic quantum field is interesting to review.

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AUTHOR CONTRIBUTIONS

AR provides research ideas, making the mathematical formulation's analytical and numerical solution.

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